Solving Parametric Polynomial Systems

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Abstract: We present a new algorithm for solving basic parametric constructible or semi-algebraic systems like $\mathcal{C} = \{x \in \mathbb{C}^n, p_1(x) = 0, \ldots, p_s(x) = 0, f_1(x) \neq 0, \ldots, f_l(x) \neq 0\}$ or $\mathcal{S} = \{x \in \mathbb{C}^n, p_1(x) = 0, \ldots, p_s(x) = 0, f_1(x) > 0, \ldots, f_l(x) > 0\}$, where $p_i, f_i \in \mathbb{Q}[U, X]$, $U = [U_1, \ldots, U_d]$ is the set of parameters and $X = [X_{d+1}, \ldots, X_n]$ the set of unknowns.

If Π_U denotes the canonical projection on the parameter's space, solving $\mathcal{C}$ or $\mathcal{S}$ remains to compute sub-manifolds $\mathcal{U} \subset \mathbb{C}^d$ (resp. $\mathcal{U} \subset \mathbb{R}^d$) such that $(\Pi_U^{-1}(\mathcal{U}) \cap \mathcal{C}, \Pi_U)$ is an analytic covering of $\mathcal{U}$ (we say that $\mathcal{U}$ has the $(\Pi_U, \mathcal{C})$-*covering property*). This guarantees that the cardinal of $\Pi_U^{-1}(\sqcap) \cap \mathcal{C}$ is locally constant on $\mathcal{U}$ and that $\Pi_U^{-1}(\mathcal{U}) \cap \mathcal{C}$ is a finite collection of sheets which are all locally homeomorphic to $\mathcal{U}$. In the case where $\Pi_U(\mathcal{C})$ is dense in $\mathbb{C}^d$, all the known algorithms for solving $\mathcal{C}$ or $\mathcal{S}$ compute implicitly or explicitly a Zariski closed subset W such that any sub-manifold of $\mathbb{C}^d \setminus W$ have the $(\Pi_U, \mathcal{C})$-covering property.

We introduce the *discriminant varieties of $\mathcal{C}$ w.r.t.* Π_U which are algebraic sets with the above property (even in the cases where Π_U is not dense in $\mathbb{C}^d$). We then show that the set of points of $\overline{\Pi_U(\mathcal{C})}$ which do not have any neighborhood with the $(\Pi_U, \mathcal{C})$-covering property is a Zariski closed set and thus the *minimal discriminant variety of $\mathcal{C}$ w.r.t.* Π_U and we propose an algorithm to compute it efficiently. Thus, solving the parametric system $\mathcal{C}$ (resp. $\mathcal{S}$) then remains to describe $\mathbb{C}^d \setminus W_D$ (resp. $\mathbb{R}^d \setminus W_D$) which can be done using critical points method or partial CAD based strategies.

We did not fully study the complexity, but in the case of systems where $\overline{\Pi_U(\mathcal{C})} = \mathbb{C}^d$, the degree of the minimal discriminant variety as well as the running time of an algorithm able to compute it are singly exponential in the number of variables according to already known results.

Key-words: Computer Algebra, Polynomial Systems, Parametric Systems, Real Roots

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Résolution des Systèmes Polynomiaux Paramètrés

Résumé : Nous présentons un nouvel algorithme pour la résolution des ensembles constructibles ou semi-algébriques basiques de la forme $\mathcal{C} = \{x \in \mathbb{C}^n, p_1(x) = 0, \dots, p_s(x) = 0, f_1(x) \neq 0, \dots, f_l(x) \neq 0\}$ ou $\mathcal{S} = \{x \in \mathbb{C}^n, p_1(x) = 0, \dots, p_s(x) = 0, f_1(x) > 0, \dots, f_l(x) > 0\}$, où $p_i, f_i \in \mathbb{Q}[U, X]$, $U = [U_1, \dots, U_d]$ est l'ensemble des paramètres et $X = [X_{d+1}, \dots, X_n]$ celui des inconnues.

Si Π_U est la projection canonique sur l'espace des paramètres, résoudre $\mathcal{C}$ ou $\mathcal{S}$ revient à calculer des sous-variétés différentiables $\mathcal{U} \subset \mathbb{C}^d$ (resp. $\mathcal{U} \subset \mathbb{R}^d$) telles que $(\Pi_U^{-1}(\mathcal{U}) \cap \mathcal{C}, \Pi_U)$ soit un revêtement analytique de $\mathcal{U}$ (on dira alors que $\mathcal{U}$ à la propriété de $(\Pi_U, \mathcal{C})$-*revêtement*). Ceci garantit que le cardinal de $\Pi_U^{-1}(\sqcap) \cap \mathcal{C}$ est localement constant sur $\mathcal{U}$ et que $\Pi_U^{-1}(\mathcal{U}) \cap \mathcal{C}$ est une collection finie de feuillets localement homéomorphes à $\mathcal{U}$. Dans le cas où $\Pi_U(\mathcal{C})$ est dense dans $\mathbb{C}^d$, les algorithmes résolvant $\mathcal{C}$ ou $\mathcal{S}$ calculent implicitement ou explicitement un fermé de Zariski W tel que toutes les sous variétés différentiables constituant $\mathbb{C}^d \setminus W$ ont la propriété de $(\Pi_U, \mathcal{C})$-revêtement.

Nous introduisons la notion de *variété discriminante de $\mathcal{C}$ relativement à* Π_U comme étant un ensemble algébrique ayant la propriété énoncée ci-dessus dans un cadre général (ne supposant pas que $\Pi_U(\mathcal{C})$ est dense dans $\mathbb{C}^d$). Nous montrons que l'ensemble des points de $\overline{\Pi_U(\mathcal{C})}$ n'ayant aucun voisinage vérifiant la propriété de $(\Pi_U, \mathcal{C})$-revêtement est un fermé de Zariski et par conséquent une *variété discriminante minimale de $\mathcal{C}$ relativement à* Π_U et nous proposons un algorithme la calculant efficacement. Ainsi, résoudre $\mathcal{C}$ $(\mathcal{S})$ revient alors à décrire $\overline{\Pi_U(\mathcal{C})} \setminus W_D$ (resp. $\mathbb{R}^d \cap (\overline{\Pi_U(\mathcal{C})} \setminus W_D)$), ce qui peut être fait en utilisant une méthode de points critiques ou encore de décomposition cylindrique algébrique partielle.

Nous n'avons pas encore étudié complètement la complexité de l'algorithme que nous proposons, mais dans le cas de systèmes où $\overline{\Pi_U(\mathcal{C})} = \mathbb{C}^d$, les résultats connus montrent que l'algorithme est simplement exponentiel en le nombre de variables.

Mots-clés : Calcul Formel, Systèmes polynomiaux, Systèmes paramétrés, Zéros réels

1 Introduction

In this article, we propose a new method for studying basic constructible (resp. semi-algebraic) sets defined as systems of equations and inequations (resp. inequalities) depending on parameters. The following notations will be used :

Notation 1 *Let us consider the basic semi-algebraic set*

$$\mathcal{S} = \{x \in \mathbb{R}^n \quad , \quad p_1(x) = 0, \ldots, p_s(x) = 0, f_1(x) > 0, \ldots f_s(x) > 0\}$$

and the basic constructible set

$$\mathcal{C} = \{x \in \mathbb{C}^n \quad , \quad p_1(x) = 0, \ldots, p_s(x) = 0, f_1(x) \neq 0, \ldots f_s(x) \neq 0\}$$

where p_i, f_j are polynomials with rational coefficients.

- $[U, X] = [U_1, \ldots U_d, X_{d+1}, \ldots X_n]$ *is the set of indeterminates or variables, while* $U = [U_1, \ldots U_d]$ *is the set of parameters and* $X = [X_{d+1}, \ldots X_n]$ *the set of unknowns;*
- $\mathcal{E} = \{p_1, \ldots p_s\}$ *is the set of polynomials defining the equations;*
- $\mathcal{F} = \{f_1, \ldots f_l\}$ *is the set of polynomials defining the inequations in the complex case or the inequalities in the real case;*
- *For any* $u \in C^d$ *,* ϕ_u *the specialization map* $U \longrightarrow u$*;*
- $\Pi_U : \mathbb{C}^n \longrightarrow \mathbb{C}^d$ *denotes the canonical projection on the parameter's space* $(u_1, \ldots, u_d, x_{d+1}, \ldots, x_n) \longrightarrow (u_1, \ldots, u_d)$*;*
- *Given any ideal I we denote by $V(I) \subset \mathbb{C}^n$ the associated (algebraic) variety. If a variety is defined as the zero set of polynomials with coefficients in $\mathbb{Q}$ we call it a $\mathbb{Q}$-algebraic variety; we extend naturally this notation in order to talk about $\mathbb{Q}$-irreducible components, $\mathbb{Q}$-Zariski closure, ..*
- *for any set $\mathcal{V} \subset \mathbb{C}^n$, $\overline{\mathcal{V}}$ will denote its $\mathbb{C}$-Zariski closure.*

Solving $\mathcal{C}$ or $\mathcal{S}$ remains to compute sub-manifolds $\mathcal{U} \subset \mathbb{C}^d$ (resp. $\mathcal{U} \subset \mathbb{R}^d$) such that $(\Pi_U^{-1}(\mathcal{U}) \cap \mathcal{C}, \Pi_U)$ is an analytic covering of $\mathcal{U}$ (in that case, we say that $\mathcal{U}$ has the $(\Pi_U, \mathcal{C})$-*covering property*). This guarantees that the cardinal of $\Pi_U^{-1}(\sqcap) \cap \mathcal{C}$ is locally constant on $\mathcal{U}$ and that $\Pi_U^{-1}(\mathcal{U}) \cap \mathcal{C}$ is a finite collection of sheets which are all locally homeomorphic to $\mathcal{U}$. In the case where $\Pi_U(\mathcal{C})$ is dense in $\mathbb{C}^d$, all the known algorithms for solving $\mathcal{C}$ or $\mathcal{S}$ compute implicitly or explicitly a Zariski closed subset W such that any sub-manifold of $\mathbb{C}^d \setminus W$ have the $(\Pi_U, \mathcal{C})$-covering property (see some examples below).

In the first section of the present article, we introduce the *discriminant varieties of $\mathcal{C}$ w.r.t.* Π_U which are algebraic sets with the above property (even in the cases where Π_U is not dense in $\mathbb{C}^d$). We then show that the set of points of $\overline{\Pi_U(\mathcal{C})}$ which do not have any neighborhood with the $(\Pi_U, \mathcal{C})$-covering property is a Zariski closed set and thus the *minimal discriminant variety of $\mathcal{C}$ w.r.t.* Π_U.

The Cylindrical Algebraic Decomposition [7] adapted to $\mathcal{E} \bigcup \mathcal{F}$ computes a discriminant variety of $\mathcal{C}$ as soon as the recursive projection steps w.r.t. $X_{d+1}, \ldots, X_n$ are done first. In fact, a discriminant variety W is defined by the union of the varieties associated with the polynomials obtained after the $n-d$-th projection step. It is far from being optimal with respect to the number of connected open sets of $\overline{\Pi_U(\mathcal{C})} \setminus W$, as it contains at least a discriminant variety for any system of equalities and inequalities that may be constructed with the polynomials of $\mathcal{E} \bigcup \mathcal{F}$. In the case of a partial CAD [6], the induced discriminant variety is smaller but depends anyway on the order in which the projections are done : it is thus not minimal in general. The same remark applies for any recursive method based on univariate resultant.

Algorithms based on *Comprehensive Gröbner bases* [24], [25], [26], compute also (implicitly or explicitly) discriminant varieties. In the case of a parametric system, such a discriminant variety contains at least the parameters' values for which a Gröbner basis do not specialize properly. Again, it is far from being optimal since it contains the parameter's values where the staircase varies, which depend on the strategy used during the computation (for example the choice of a monomial ordering) and which may not belong to the minimal discriminant variety.

Methods that computes parameterizations of the solutions (see [21] for example) compute also discriminant varieties. Again, these are not optimal since the result depends on the strategy which is used. Precisely, it depends on an arbitrary chosen \"separating element \" $t \in \mathbb{Q}[X_{d+1}, \ldots, X_n]$ which defines a polynomial (usually linear) application from $\phi_u(V(\mathcal{E}))$ to $\mathbb{C}$ which is injective for almost all specializations of the parameters. If p is the minimal polynomial of t viewed as an element of $\mathbb{Q}(U_1, \ldots, U_d)[X_{d+1}, \ldots, X_n]/\sqrt{<\mathcal{E}>}$, the induced discriminant variety contains all the parameters' values such that the leading coefficient of p or its discriminant vanishes; this includes, in particular, the parameter's values such that t does not separate the zeros of the corresponding specialization of $\sqrt{\mathcal{E}}$.

Conversely, our minimal discriminant variety is an intrinsic object, which is defined independently from any computational strategy.

In the second part of the present paper, we propose an algorithm for computing the minimal discriminant variety of a basic constructible set w.r.t a given projection, by using exclusively well known tools such as efficient Gröbner bases [10], triangular sets [2] (for some exceptional bad situations), and some variants of the Rational Univariate Representation [17]. In our algorithm, the computations depends strongly on the properties of the ideal $<\mathcal{E}>$ (equi-dimensional or not, radical or not, etc.) : it is very efficient for most systems coming from applications but it is also able to detect and to solve all the worst cases : simple tests will drive the computations in every situation.

The third part deals with the use of discriminant varieties for solving several problems like computing the number of real roots w.r.t. parameter's values, or to provide a meaningful decomposition of $\mathcal{S}$. Once a discriminant variety W is known, it is sufficient to describe all the semi-algebraically connected components of $\overline{\Pi_U(\mathcal{C})} \setminus W$ to solve many problems. For example, if we take one point u in any of these components $\mathcal{U}$ (see [3],[19],[20],[4] for example) one can fully describe $\Pi_U^{-1}(\mathcal{U}) \cap \mathcal{C}$ by solving the zero-dimensional system $\Phi_u(\mathcal{C})$ (using [17], [11] or [22] for example). For some applications, one need to describe more precisely $\overline{\Pi_U(\mathcal{C})} \setminus W$: this can be done by using the partial CAD algorithm such as in [19, 6]. Ad-hoc versions of our algorithm were already used in [13, 14, 8] to solve several practical problems. They can be viewed as instances of the method described in this paper, and will serve as examples for illustrating various technical points.

Complexity issues have not been fully addressed for the moment, but a straightforward analysis using existing results like [12], in the particular case of systems where $\overline{\Pi_U(\mathcal{C})} = \mathbb{C}^d$, shows that the computation time of a large discriminant variety (and thus its size) is singly exponential in the number of unknowns.

2 The discriminant variety

Let's start with a precise definition of a discriminant variety of $\mathcal{C}$ w.r.t. Π_U :

Definition 1 *Let $\Pi = \overline{\Pi_U(\mathcal{C})} = \overline{\Pi_U(\overline{\mathcal{C}})}$ and δ be the dimension of Π. An algebraic variety W is a discriminant variety of $\mathcal{C}$ w.r.t. Π_U iff:*

- *W is contained in Π;*
- *$W = \Pi$ iff $\Phi_u(\mathcal{C})$ is infinite or empty for almost all $u \in \Pi$;*
- *The connected components $\mathcal{U}_1, \ldots \mathcal{U}_k$ of $\Pi \setminus W$ are analytic sub-manifolds of dimension δ (If Π is connected, there is only one component).*
- *For $i = 1 \ldots k$, $(\Pi_U^{-1}(\mathcal{U}_i) \bigcap \mathcal{C}, \Pi_U)$ is an analytic covering of $\mathcal{U}_i$.*

If W is a discriminant variety, $(\Pi_U^{-1}(\mathcal{U}) \bigcap \mathcal{C}, \Pi_U)$ is an analytic covering of $\mathcal{U}$ for $\mathcal{U} \in \{\mathcal{U}_1 \ldots \mathcal{U}_k\}$, which implies that:

- there exists a finite set of indexes $\mathcal{I}$ and disjoints connected sub-sets $(\mathcal{V}_i)_{i \in \mathcal{I}}$ of $\mathcal{C}$ such that $\Pi_U^{-1}(\mathcal{U}) \bigcap \mathcal{C} = \bigcup_{i \in \mathcal{I}} \mathcal{V}_i$;
- Π_U is a local diffeomorphism from $\mathcal{V}_i$ to $\mathcal{U}$;

Since $\mathcal{C}$ is a constructible set, for any $u \not\in W$, the discrete set $\Pi_U^{-1}(u) \bigcap \mathcal{C}$ is necessarily finite. In particular, W contains the projection of every component of dimension $> \delta$ of $\mathcal{C}$.

If O_{sd} is the projection of the irreducible components of $\overline{\mathcal{C}}$ of dimensions $< \delta$, then O_{sd} is obviously contained in W.

If $\mathcal{O}_\infty$ is the set of the $u \in \Pi$ such that $\Pi_U^{-1}(\mathcal{U}) \bigcap \overline{\mathcal{C}}$ is not compact for any compact neighborhood $\mathcal{U}$ of u, then $\mathcal{O}_\infty \subset W$. In fact, if $\mathcal{U} \subset \Pi \setminus W$ is a compact neighborhood of a point of $\Pi \setminus W$, then $\Pi_U^{-1}(\mathcal{U}) \bigcap \overline{\mathcal{C}}$ is compact since the restriction of Π_U on each $\mathcal{V}_i$ is a local diffeomorphism.

If O_c is the set of critical values of Π_U, O_c it is also contained in W since the restriction of Π_U to $\mathcal{V}_i$ is a local diffeomorphism. On may notice that the critical values of Π_U on the components of $\overline{\mathcal{C}}$ of dimension $\neq \delta$ are contained in $O_{sd} \cup O_\infty$. Thus, one may restrict O_c to the critical values of the restriction of Π_U to the union of the components of dimension δ.

If $x \in \overline{\mathcal{C}} \setminus \mathcal{C}$, it belongs to the closure of $\mathcal{C}$ for the usual topology, then $\Pi_U(x) \in W$ since Π_U is a local diffeomorphism.

Finally, if W_{sing} is the singular locus of Π, then, by definition, $W_{sing} \subset W$. One may notice that in many applications, $d = \delta$, which means that $\Pi = \mathbb{C}^d$ and implies $W_{sing} = \emptyset$.

The following lemma summarizes these properties and definitions:

Lemma 1 *Let $\mathcal{C}$ be a constructible set defined as in Notation 1 and $\overline{\mathcal{C}}$ be its $\mathbb{Q}$-Zariski closure in $\mathbb{C}^n$. Let us define:*

- *O_{sd} the projection of the irreducible components of $\overline{\mathcal{C}}$ of dimension less than δ;*
- *O_c the critical values of Π_U in restriction to the union of the components of dimension δ of $\overline{\mathcal{C}}$;*
- *O_∞ the set of points $u \in \Pi$ such that $\Pi_U^{-1}(\mathcal{U}) \bigcap \overline{\mathcal{C}}$ is not compact for any compact neighborhood $\mathcal{U}$ of u in Π;*
- *$O_\mathcal{F}$ (resp. O_{F_i}) the projection of the intersection of $\overline{\mathcal{C}}$ with the hyper-surface defined by $\prod_{i=1}^s f_i$ (resp. by f_i);*
- *W_{sing} the singular points locus of $\overline{\Pi_U(\mathcal{C})}$ (which is a $\mathbb{Q}$-variety).*

If W is a discriminant variety of $\mathcal{C}$ w.r.t. Π_U, then $O_{sd} \cup O_c \cup O_\infty \cup O_\mathcal{F} \cup W_{sing} \subset W$.

The main goal in the rest of this section to show that $O_{sd} \cup O_c \cup O_\infty \cup O_\mathcal{F} \cup W_{sing}$ is a discriminant variety, and thus the smallest one. In addition, we will give an algebraic characterization of this smallest discriminant variety, making possible its computation as described in the next section. Finally, we will show that this minimal discriminant variety has a dimension smaller than the dimension δ of Π if and only if the irreducible components of dimension $> \delta$ of $\overline{\mathcal{C}}$ have a projection by Π_U of dimension $< \delta$.

We first show that $W_{sing} \cup O_{sd} \cup O_c \cup O_\infty \cup O_\mathcal{F}$ is a $\mathbb{Q}$-algebraic variety.

Lemma 2 *The set O_∞ is $\mathbb{Q}$-Zariski closed. More precisely it is equal to $W_\infty := \pi(\overline{\mathcal{C}}^p \bigcap \mathcal{H}_\infty)$, where:*

- *$\mathbb{P}^{n-d}$ is the projective space associated to $\mathbb{C}^{n-d}$;*
- *$\overline{\mathcal{C}}^p$ is the projective closure of $\mathcal{C}$ in $\mathbb{C}^d \times \mathbb{P}^{n-d}$;*
- *$\mathcal{H}_\infty$ is the hyper-plane at infinity in $\mathbb{C}^d \times \mathbb{P}^{n-d}$, i.e. $\mathcal{H}_\infty = \left(\mathbb{C}^d \times \mathbb{P}^{n-d}\right) \setminus \left(\mathbb{C}^d \times \mathbb{C}^{n-d}\right)$;*
- *π is the canonical projection from $\mathbb{C}^d \times \mathbb{P}^{n-d}$ to $\mathbb{C}^d$.*

Proof According to [9] (corollary 10 p. 389), if $\mathcal{C} \subset \mathbb{C}^n$ is any constructible set, then $\overline{\Pi_U(\overline{\mathcal{C}})} = \overline{\Pi_U(\mathcal{C})} = \pi(\overline{\mathcal{C}}^p)$. Since $\overline{\mathcal{C}}$ is the affine part of $\overline{\mathcal{C}}^p$, then:

$$\overline{\Pi_U(\mathcal{C})} = W_\infty \bigcup \Pi_U(\overline{\mathcal{C}}) \qquad (*).$$

According to [16], W_∞ is a $\mathbb{C}$-algebraic variety since it is the projection on the affine space $\mathbb{C}^d$ of a $\mathbb{C}$-variety of $\mathbb{C}^d \times \mathbb{P}^{n-d}$. Moreover, it is a $\mathbb{Q}$-variety since it can be written as the intersection of $\mathbb{Q}$-varieties.

Let $u \in \Pi$.

If $u \notin W_\infty$, then according to (*), there exists a compact neighborhood $\mathcal{U} \subset \Pi_U(\overline{\mathcal{C}})$ of u such that $\mathcal{U} \bigcap W_\infty = \emptyset$, and thus $\Pi_U^{-1}(\mathcal{U}) \bigcap \overline{\mathcal{C}} = \pi^{-1}(\mathcal{U}) \bigcap \overline{\mathcal{C}}^p$. Since π is continuous $\Pi_U^{-1}(\mathcal{U}) \bigcap \overline{\mathcal{C}}$ is then compact, which shows that $u \notin O_\infty$ and $O_\infty \subset W_\infty$.

On the other hand, if u belongs to W_∞, there exists, by definition of W_∞, an element t of $\mathbb{P}^{n-d}$ such that $(u, t) \in \overline{\mathcal{C}}^p \bigcap \mathcal{H}_\infty$. By definition of $\overline{\mathcal{C}}^p$, any neighborhood of (u, t) in $\mathbb{C}^d \times \mathbb{P}^{n-d}$ meets $\overline{\mathcal{C}}$, which implies that the reciprocal image by Π_U of any compact neighborhood of u intersects $\overline{\mathcal{C}}$ and is not compact since it is different from its closure in $\mathbb{C}^d \times \mathbb{P}^{n-d}$. Thus $W_\infty \subset O_\infty$. $\square$

The sets O_{sd} and O_c are not Zariski closed in general, but they are projections of $\mathbb{Q}$-Zariski closed subsets of $\overline{\mathcal{C}}$. Thus, according to the relation (*), we have $(\overline{O_{sd}} \setminus O_{sd}) \subset W_\infty$ and $(\overline{O_c} \setminus O_c) \subset W_\infty$. This shows:

Lemma 3 *$O_{sd} \bigcup O_c \bigcup O_\infty$ is $\mathbb{Q}$-Zariski closed. More precisely, if W_{sd}, (resp. W_c) denotes the $\mathbb{Q}$-Zariski closure of O_{sd} (resp. O_c), then $O_{sd} \bigcup O_c \bigcup O_\infty = W_{sd} \bigcup W_c \bigcup W_\infty$*

By definition, W_{sing}, W_{sd} and $W_c = \overline{\mathcal{O}}_\rfloor$ are closed subsets of dimension $< \delta$, so that $W_{sing} \bigcup O_{sd} \bigcup O_c \bigcup O_\infty$ is a closed set which is strictly contained in Π if and only if $W_\infty = O_\infty$ is strictly contained in Π.

Adding the inequations to the problem in the complex case or the inequalities in the real case can be done in a simple way. If $\mathcal{D}$ is a connected component of $\overline{\mathcal{C}}$, a polynomial $f_i \in \mathcal{F}$ can not be identically null on $\mathcal{D}$ (by definition of $\mathcal{C}$). Thus $\overline{\Pi_U(\mathrm{V}(F_i) \bigcap \overline{\mathcal{C}})}$ is a strict subset of Π. Using again (*), $O_\mathcal{F}$ is the projection of an algebraic set contained in $\overline{\mathcal{C}}$ and $(\overline{O_\mathcal{F}} \setminus O_\mathcal{F}) \subset W_\infty$. Setting $W_\mathcal{F} = \overline{O_\mathcal{F}}$ and $W_{F_i} = \overline{O_{F_i}}$, we obtain the following result:

Lemma 4 *The set $O_\infty \bigcup O_\mathcal{F}$ is $\mathbb{Q}$-Zariski closed and is contained in every discriminant variety of $\mathcal{C}$. Therefore, $W_{sing} \bigcup O_{sd} \bigcup O_c \bigcup O_\infty \bigcup O_\mathcal{F} = W_{sing} \bigcup W_{sd} \bigcup W_c \bigcup W_\infty \bigcup W_\mathcal{F}$ is also $\mathbb{Q}$-Zariski closed.*

According to this lemma, we have defined a $\mathbb{Q}$-algebraic variety that is included in any discriminant variety. It remains to show that this object is itself a discriminant variety:

Theorem 1 *$W_{sing} \bigcup O_{sd} \bigcup O_c \bigcup O_\infty \bigcup O_\mathcal{F} = W_{sing} \bigcup W_{sd} \bigcup W_c \bigcup W_\infty \bigcup W_\mathcal{F}$ is the smallest discriminant variety of $\mathcal{C}$ w.r.t. Π_U.*

Proof The only thing that remains to proof is that for any point u in the complementary of $W_D = W_{sing} \bigcup W_{sd} \bigcup W_c \bigcup W_\infty \bigcup W_\mathcal{F}$ in Π, there exists a sub-manifold $\mathcal{U} \subset \Pi$ of dimension δ containing u and such that $(\Pi_U^{-1}(\mathcal{U}) \bigcap \mathcal{C}, \Pi_U)$ is an analytic covering of $\mathcal{U}$.

If $W_\infty = \Pi$, then $W_D = \Pi$ and the lemma is proved. Let us now suppose that $W_\infty \neq \Pi$. In this case, W_D is strictly contained in Π (the other components of W_D have dimension $< \delta$).

By definition $W_\infty \subset W_D$, and thus $\Pi_U^{-1}(u) \bigcap \mathcal{C}$ is a non empty compact set for any $u \notin W_D$. It is therefore finite. More generally, by continuity of Π_U, if $\mathcal{U}$ is a compact neighborhood of u in Π which do not meet W_D, $\Pi_U^{-1}(\mathcal{U}) \bigcap \overline{\mathcal{C}}$ is compact. Since $O_{sing} \subset W_D$, there always exists a neighborhood $\mathcal{U}$ of u contained in Π_U that is a sub-manifold of dimension δ.

Let u be a point in $\Pi \setminus W_D$ and $\mathcal{U}$ a compact neighborhood of u such that $\mathcal{U} \bigcap W_D = \emptyset$. Let $\mathcal{D}$ be a connected component of $\Pi_U^{-1}(\mathcal{U}) \bigcap \mathcal{C}$. Since $\mathcal{D}$ is compact, if $\mathcal{D}$ do not meet $\Pi_U^{-1}(u)$, we can restrict $\mathcal{U}$ to a sub-manifold $\mathcal{U}' \subset \mathcal{U}$ containing u and such that $\Pi_U^{-1}(\mathcal{U}') \bigcap \mathcal{D} = \emptyset$. Similarly, we can suppose that all the connected components of $\Pi_U^{-1}(\mathcal{U}) \bigcap \mathcal{C}$ intersect $\Pi_U^{-1}(u)$. Since $u \notin O_\infty \bigcup O_{sd}$, these components have dimension δ. Since $u \notin \mathcal{O}_c$, the implicit functions theorem applies. After having possibly reduced $\mathcal{U}$, Π_U then defines a $\mathbb{C}^\infty$-diffeomorphism between each of these connected components and $\mathcal{U}$ and thus, $(\Pi_U^{-1}(\mathcal{U}), \Pi_U)$ is an analytic covering of $\mathcal{U}$ ([5]). $\square$

Definition 2 *The minimal discriminant variety of $\mathcal{C}$ w.r.t Π_U is the $\mathbb{Q}$-algebraic variety $W_D = W_{sing} \bigcup O_{sd} \bigcup O_c \bigcup O_\infty \bigcup O_\mathcal{F}$.*

In the real case the restriction to $\mathbb{R}^n$ of the covering induced by the discriminant variety is a real analytic covering. Thus the polynomials of $\mathcal{F}$ have constant sign and do not vanish on the connected component of this covering.

Remark 1 One must note that we may have $W_\infty = \Pi$ from a complex point of view while the systems has a finite number if real roots for almost all the admissible parameter's values. This is the case if there are complex components of dimension $> \delta$ without any real point.

In the same spirit, the discriminant variety defined as above is optimal for the number of sub-manifolds induced in the parameter's space, but this is not true in the real case : it may exists a sub-manifold $\mathcal{U} \subset \Pi$ of dimension δ which meets W_D but such that $(\Pi_U^{-1}(\mathcal{U}) \bigcap \mathcal{S}, \Pi_U)$ is an analytic covering of $\mathcal{U}$; This may be the case if some real sheets of W_D correspond to purely complex events.

Therefore, it is not possible to define a real minimal discriminant variety (which is an algebraic set). The best which may be done is to define a *discriminant semi-algebraic set* which is the union of the real connected components of W_D which are the projections of real critical points, points at infinity, components of smaller dimension, etc. We do not define this formally because we do not know a better way to compute it than to compute W_D and to study the topology of its real components.

It is sometimes useful to replace the conditions $F_i > 0$ or $F_i \neq 0$ by \"F_i has a constant sign on each sheet of the covering \". This may be done by a slight modification of the definition of the discriminant variety:

Definition 3 *Let $V = V(<\mathcal{E}>)$. If $\Pi_U(V)$ and $\Pi_U(\mathcal{C})$ have the same dimension, we define the minimal discriminant variety of V w.r.t. Π_U and adapted to $\mathcal{F}$ as the union of the minimal discriminant variety of V w.r.t. Π_U and of the component $W_{\mathcal{F}}$ of the minimal discriminant variety of $\mathcal{C}$ w.r.t. Π_U.*

One may notice that this discriminant variety may be different from the discriminant variety of $\mathcal{C}$ if V has some component of dimension $\geq \delta$ which is contained in the zero set of some F_i, i.e. if $V \neq \overline{\mathcal{C}}$. This discriminant variety has the following property:

Corollary 1 *If W is the discriminant variety of V w.r.t Π_U and adapted to $\mathcal{F}$, then its complement in Π is a union of manifolds $\mathcal{U}_1, \ldots \mathcal{U}_k$ of dimension δ such that*

- $(\Pi_U^{-1}(\mathcal{U}_i) \bigcap V, \Pi_U)$ *is an analytic covering of $\mathcal{U}_i$ for all $i = 1 \ldots k$;*
- *the elements of $\mathcal{F}$ have constant signs on the connected components of the real part of $\Pi_U^{-1}(\mathcal{U}_i) \bigcap V$, for all $i = 1 \ldots k$;*

Let us end this section by some remarks about the dimension of the discriminant varieties.

Lemma 5 *Let W (resp. W_D) be a discriminant variety (resp. the minimal discriminant variety) of $\mathcal{C}$ w.r.t. Π_U such that $W_D \neq \Pi$. If Π is irreducible, then $\dim(W_D) < \delta$ and $\dim(\overline{W \setminus W_D}) < \delta$. If Π is not irreducible, one may have $\dim(W_D) = \delta$ or/and $\dim(\overline{W \setminus W_D}) = \delta$.*

Proof The first item is a direct consequence of the definition 1. For the second one, consider the following algebraic variety in $\mathbb{C}^3$: $V = \{U^2+V^2-1=0\} \cup \{(U-3)^2+(V-3)^2-1=0, X=1\} \cup \{(U+3)^2+(V+3)^2-1=0, X=1\}$. If the parameters are U and V, then $\Pi = \{U^2+V^2-1=0\} \cup \{(U-3)^2+(V-3)^2-1=0\} \cup \{(U+3)^2+(V+3)^2-1=0\}$, $W_\infty = W_D = \{U^2+V^2-1\}$ and $W = \{U^2+V^2-1=0\} \cup \{(U-3)^2+(V-3)^2-1=0\}$ is a (non minimal) discriminant variety. □

3 Computing discriminant varieties

In this section, we give a general algorithm for computing the minimal discriminant variety of any basic constructible set. Given any ideal $I \subset \mathbb{Q}[U, X]$ such that $\mathrm{V}(I) = \overline{\mathcal{C}}$, we will first recall how to compute d, δ, $I \cap \mathbb{Q}[U]$ and we will show how to compute explicitly the generators of ideals $I_{\mathcal{F}}, I_\infty \subset \mathbb{Q}[U]$ such that $\mathrm{V}(I_\infty) = W_\infty$ and $\mathrm{V}(I_{\mathcal{F}}) = W_{\mathcal{F}}$ without any assumption on $\mathcal{E}$. The computation of the other components of W_D or of any discriminant variety depends strongly on the properties of I. Also before going on with the algorithmic part, let's precise some of our goals and targets:

3.1 Some remarks and key targets

Once W_∞ and $W_{\mathcal{F}}$ are known, a difficult task is the computation of $W_c, W_{\mathrm{sd}}, W_{\mathrm{sing}}$. One main target is to avoid when possible costly computations such as decomposing I (as intersection of equi-dimensional or primary ideals) or computing its radical. Lets consider the case of W_c for example; If I is prime, then W_c is the zero set of $(I + \mathrm{Jac}_X^{n-\delta}(I)) \cap \mathbb{Q}[U]$ where $\mathrm{Jac}_X^{n-\delta}(I)$ is the ideal generated by all the minors $n-\delta$ of the Jacobian matrix with respect to the variables X of any systems of generators of I. This characterization can be extended to equi-dimensional and radical ideals but not to the general case (consider for example the system $P^2 = 0$ where P is a non constant polynomial in $\mathbb{Q}[U, X]$).

For many parametric systems coming from applications, $\Phi_u(\mathcal{E})$ can be numerically solved for almost all $u \in \mathbb{C}^d$ using simple versions of Newton's algorithm. This means in particular that $d = \delta, s = n - \delta$ and that $<\Phi_u(\mathcal{E})>$is radical and zero-dimensional for almost all $u \in \mathbb{C}^d$. For such class of systems, $<\mathcal{E}>$ may be not radical or/and not equi-dimensional, but we always have $W_{\mathrm{sd}} = W_{\mathrm{sing}} = \emptyset$. A consequence of some results presented in this section is that even if $W_c \subsetneqq (I + \mathrm{Jac}_X^{n-\delta}(I)) \cap \mathbb{Q}[U]$ for such systems, we always have $W_D = W_{\mathcal{F}} \cup W_\infty \cup \mathrm{V}(I + \mathrm{Jac}_X^{n-\delta}(I)) \cap \mathbb{Q}[U])$

so that there is no need to decompose I or to compute its radical. More generally, we will characterize a class of systems for which $W_D = W_{\mathcal{F}} \cup W_\infty \cup \mathrm{V}(I + \mathrm{Jac}_X^{n-\delta}(I)) \cap \mathbb{Q}[U]) \cup \mathrm{V}(I \cap \mathbb{Q}[U] + \mathrm{Jac}_X^{n-\delta}(I \cap \mathbb{Q}[U]))$ and propose an algorithm that first check if a problem belongs to this class and, if so, computes its minimal discriminant variety.

If the algorithm concludes that the problem doesn't belong to a favorable class, there are many situations where $W' = W_{\mathcal{F}} \cup W_\infty \cup \mathrm{V}(I + \mathrm{Jac}_X^{n-\delta}(I)) \cap \mathbb{Q}[U]) \cup \mathrm{V}(I \cap \mathbb{Q}[U] + \mathrm{Jac}_X^{n-\delta}(I \cap \mathbb{Q}[U]))$ is a large discriminant variety or a large discriminant variety where components of W_{sd} are missing. Even if some components of W_{sd} are missing, this variety may be an acceptable answer for some applications : W' is then a discriminant variety of the union of the components of dimension $\geq \delta$ of $\overline{\mathcal{C}}$. Also, over each connected open subset of $\mathcal{U} \subset \Pi \setminus W'$, the number of solutions is constant for all the parameters of $\mathcal{U}$ that do not belong to W_{sd} (and thus for almost all the parameters of $\mathcal{U}$). For many applications, this information is sufficient and there is no need to compute a discriminant variety. In section 5.1, for example, the parameters modelize the lengths of some physical components of a robot : because of unavoidable manufacturing errors, it doesn't make sense to study the case where they belong to a strict Zariski closed subset of the parameter's space.

Also, there are only few practical cases where the computation of a decomposition of I or of its radical may be useful. In fact, we will show that we only need to be able to replace some of its primary components of dimension δ (in a minimal primary decomposition) by their radical, remove the primary components of dimension $< \delta$ embedded in primary components of dimension δ and compute the intersection of primary components of dimension $< \delta$. Even if it does much more job than needed, we have chosen to present a strategy based on triangular sets when W' is not a satisfactory output.

3.2 Conditions free computations

Most of the components of the minimal discriminant variety are the $\mathbb{Q}$-Zariski closures of the projection by Π_U of some algebraic variety V; if I is such that $\mathrm{V}(I) = V$, then $\overline{\Pi_U(V)} = \mathrm{V}(I \bigcap \mathbb{Q}[U])$. If G is a Gröbner basis of I for a monomial ordering which eliminate X, then $G \bigcap \mathbb{Q}[U]$ is a Gröbner basis of $I \bigcap \mathbb{Q}[U]$, and this is the simplest way to compute $\overline{\Pi_U(V)}$. In practice, the most efficient monomial ordering is a block ordering which is the Degree Reverse Lexicographic one (DRL) on each block. As we will need more properties of such block orderings, we have to precise the notations:

Notation 2 *Let $U = [U_1, \ldots, U_d] \subset [Y_1, \ldots Y_n]$ and $X = [X_{d+1}, \ldots, X_n] = Y \setminus U$. If $<_U$ (resp. $<_X$) is an admissible monomial ordering for the monomials depending on the variables U (resp. X), $<_{U,X} = (<_U, <_X)$ will denote the product of orderings such that $U_i <_{U,X} X_i$ for $U_i \in U$ and $X_i \in X$.*

Given any admissible ordering $<$ on a subset of the set Y of the variables, and any polynomial $g \in \mathbb{Q}[Y]$, $LM_<(g)$ (resp. $LC_<(g)$), will denote the leading monomial (resp. the leading coefficient) of g with respect to $<$. Note that with this notation, $LC_{<_X}(g)$ is a polynomial in U.

According to [9] we have:

Proposition 1 *Let G be a Gröbner basis of any ideal $I \subset \mathbb{Q}[U, X]$ w.r.t. $<_{U,X}$, then $G \bigcap \mathbb{Q}[U]$ is a Gröbner basis of $I \bigcap \mathbb{Q}[U]$ w.r.t. $<_U$;*

Let T be a new indeterminate, then $\overline{\mathrm{V}(I) \setminus \mathrm{V}(f)} = \mathrm{V}((I + \langle Tf - 1\rangle) \bigcap \mathbb{Q}[U, X])$. If $G' \subset \mathbb{Q}[U, X, T]$ is a Gröbner basis of $I + \langle Tf - 1\rangle$ with respect to $<_{(U,X),T}$ then $G' \bigcap \mathbb{Q}[U, T]$ is a Gröbner basis of $I : f^\infty := (I + \langle Tf - 1\rangle) \bigcap \mathbb{Q}[U, X]$ w.r.t. $<_{(U,X)}$. The variety $\overline{\mathrm{V}(I) \setminus \mathrm{V}(f)}$ and the ideal $I : f^\infty$ are usually called the localization of $\mathrm{V}(I)$ and I by f.

These well known results reduce the computation of $\overline{\mathcal{C}}$, Π, δ and $W_{\mathcal{F}}$ to single Gröbner bases computation for block orderings: The ideal I such that $\mathrm{V}(I) = \overline{\mathcal{C}}$ is the localization of $\langle \mathcal{E} \rangle$ by $\prod_{i=1}^l f_i$ or successively by each f_i. We can represent Π as $\mathrm{V}(I \bigcap \mathbb{Q}[U])$, and its dimension δ is easily deduced in practice from the Gröbner basis, although this is a NP-complete problem [9]. Finally; $W_{\mathcal{F}} = (I + \langle \prod_{i=1}^l f_i \rangle) \bigcap \mathbb{Q}[U]$ or equivalently $W_{\mathcal{F}} = (< \mathcal{E} >: (\prod_{i=1}^l f_i) + \langle \prod_{i=1}^l f_i \rangle) \bigcap \mathbb{Q}[U]$.

Let us propose some remarks about the computation of $W_{\mathcal{F}}$ and I:

Remark 2 The computation of$< \mathcal{E} >: (\prod_{i=1}^l f_i)$ can be avoid in several situations :

- if $< \mathcal{E} > + < \prod_{i=1}^{l} f_i >$has dimension $< \delta$: in such cases the irreducible components of $\mathrm{V}(< \mathcal{E} >)$ that belong to $\mathrm{V}(\prod_{i=1}^{l} f_i)$ are necessarily of dimension $< \delta$ and so belong to $\Pi_U^{-1}(W_{\mathrm{sd}})$.
- if $(< \mathcal{E} > + < \prod_{i=1}^{l} f_i >) \cap \mathbb{Q}[U] =< \mathcal{E} > \cap \mathbb{Q}[U]$ (which can easily be tested when knowing a Gröbner of both ideals for the same monomial ordering) then $W_D = \Pi$ (the system has no solutions).

Also, we propose the following algorithm in order to compute δ, $W_{\mathcal{F}}$, and an ideal I whose zero set coincides with $\mathcal{C}$ over $\Pi \setminus W_D$ and such that its minimal discriminant variety is W_D ($\mathrm{V}(I)$ may differ from $\overline{\mathcal{C}}$ but the difference is included in irreducible components of $\overline{\mathcal{C}}$ of dimension $< \delta$ contained in $\mathrm{V}(\prod_{i=1}^{l} f_i)$):

Algorithm PREPROCESSING

- **Input** : $\mathcal{E}, \mathcal{F}, U, X$
- Output : δ and $I, I_\Pi, I_{\mathcal{F}}$ such that
  - I a reduced Gröbner basis for $<_{U,X}$such that $\mathcal{C} \cap \Pi_U^{-1}(\Pi \setminus W_D) = \mathrm{V}(I) \cap \Pi_U^{-1}(\Pi \setminus W_D)$;
  - $I_\Pi, I_{\mathcal{F}}$ are reduced Gröbner bases for $<_U$ such that $\mathrm{V}(I_\Pi) = \Pi$ and $\mathrm{V}(I_{\mathcal{F}}) = W_{\mathcal{F}}$;
- **Begin**
  - **Compute** $G_{\mathcal{E}}$ the reduced Gröbner basis of $\mathcal{E}$ for $<_{U,X}$
  - **Deduce** $G_{\mathcal{E},U} = G_{\mathcal{E}} \cap \mathbb{Q}[U]$
  - **Compute** $d_{\mathcal{E},U}$, the dimension of $G_{\mathcal{E},U}$
  - **Compute** $G_{\mathcal{E}\cap\mathcal{F}}$, the reduced Gröbner basis of $\mathcal{E} \cup \mathcal{F}$ for $<_{U,X}$
  - **if** $(G_{\mathcal{E}\cap\mathcal{F}} = G_{\mathcal{E}})$ **then return** $(\delta = d_{\mathcal{E},U}, G_{\mathcal{E}}, I_\Pi =< G_{\mathcal{E},U} >, I_{\mathcal{F}} = I_\Pi)$
  - **else**
    - **Deduce** $G_{\mathcal{E}\cap\mathcal{F},U} = G_{\mathcal{E}\cap\mathcal{F}} \cap \mathbb{Q}[U]$
    - **Compute** $d_{\mathcal{E}\cap\mathcal{F},U}$, the dimension of $G_{\mathcal{E}\cap\mathcal{F},U}$
    - **if** $(d_{\mathcal{E},U} = d_{\mathcal{E}\cap\mathcal{F},U})$ **then**
      - **Compute** $G_{\mathcal{E},T\mathcal{F}}$, the reduced Gröbner basis of $\mathcal{E} \cup \{T(\prod_{i=1}^{l} f_i) - 1\}$ for $<_{T,(U,X)}$
      - **Deduce** $G_{\mathcal{E}:\mathcal{F}} = F_{\mathcal{E},T\mathcal{F}} \cap \mathbb{Q}[X,U]$
      - **return**(PREPROCESSING$(G_{\mathcal{E},\mathcal{F}}, \mathcal{F}, U, X)$)
    - **else return**($\delta = d_{\mathcal{E},U}$, $I = G_{\mathcal{E}}$, $I_\Pi =< G_{\mathcal{E},U} >$, $I_{\mathcal{F}} =< G_{\mathcal{E}\cap\mathcal{F},U} >$)
- **End**

We are now going to prove that we can represent W_∞ as the zeros of some Gröbner bases, which may be extracted without any further computation from the Gröbner basis w.r.t. some block ordering of any ideal I such that $\mathrm{V}(I) = \overline{\mathcal{C}}$.

Theorem 2 *Let G be a reduced Gröbner basis of any ideal I such that $V(I) = \overline{\mathcal{C}}$ for $<_{U,X}$ where $<_X$ is the Degree Reverse Lexicographic ordering s.t. $X_{d+1} < \ldots < X_n$. We define $\mathcal{E}_i^\infty = \{LC_{<_X}(g) \quad | \quad g \in G \quad , \quad \exists m \geq 0, \quad LM_{<_X}(g) = X_i^m\}$, and $\mathcal{E}_0 = G \bigcap \mathbb{Q}[U]$. Then:*

- *$\mathcal{E}_0$ is a Gröbner basis of $I \bigcap \mathbb{Q}[U]$ w.r.t. $<_U$ and $\mathcal{E}_0 \subset \mathcal{E}_i^\infty$ for $i = d+1 \ldots n$;*
- *$\mathcal{E}_i^\infty$ is a Gröbner basis of some ideal $I_i^\infty \subset \mathbb{Q}[U]$ w.r.t. $<_U$;*
- *$W_\infty = \bigcup_{i=d+1}^{n} V(I_i^\infty)$.*
- *if $I \bigcap \mathbb{Q}[U]$ is prime, then $W_\infty = \Pi$ if and only if $\mathcal{E}_i^\infty = \mathcal{E}_0$ for some i.*

Proof The first item is obvious from the definition of $\mathcal{E}_0$ and $\mathcal{E}_i^\infty$ and according to proposition 1.

Let $p \in \mathbb{Q}[U,X]$. We say that p is X-homogeneous of degree k if $p = \sum_{|\alpha|=k} h_\alpha(U)X^\alpha$. Let T be a new variable. We define the (X,T)-homogenization of p as being the (X,T)-homogeneous polynomial $p^h = \mathbb{Q}[T,U,X]$ of degree $degree(p,X)$ such that $p^h(U,X,1) = p$. By extension, if G is a set of polynomials of $\mathbb{Q}[U,X]$, G^h is the set of (X,T)-homogenizations of the elements of G and if I is an ideal of $\mathbb{Q}[U,X]$, I^h is the (X,T)-homogeneous ideal generated by the (T,X)-homogenizations of polynomials of I. If G is a Gröbner basis of I for $<_{U,X}$ then G^h is a Gröbner basis of I^h for the ordering $<_h$ such that $U^{\alpha_1}X^{\beta_1}T^{\gamma_1} <_h U^{\alpha_2}X^{\beta_2}T^{\gamma_2}$ iff ($\gamma_1 = \gamma_2$ and $U^{\alpha_1}X^{\beta_1} <_{U,X} U^{\alpha_2}X^{\beta_2}$) or ($\gamma_1 < \gamma_2$) and we have: $\overline{\mathrm{V}(I)}^p = \mathrm{V}(G^h) = \mathrm{V}(I^h)$ ([9] theorem 4 p. 375).

We need also to consider the specialization map:

$$\begin{array}{cccc} \Psi_j^a: & \mathbb{Q}[T,U,X] & \longrightarrow & \mathbb{Q}[U, X_{j+1}, \ldots X_n] \\ & T & \mapsto & 0 \\ & X_{d+1} & \mapsto & 0 \\ & \vdots & & \vdots \\ & X_{j-1} & \mapsto & 0 \\ & X_j & \mapsto & 1 \end{array}$$

The definition of the Degree Reverse Lexicographic ordering makes almost immediate the following proposition:

Lemma 6 *Let $g \in \mathbb{Q}[U,X]$; then :*

- *The ordering w.r.t. $<_{U,X}$ of the monomials in g is the same as the ordering of their images in $\Psi_j^a(g^h)$, the monomials with a null image being the smallest ones in g.*
- *If $\Psi_j^a(LM_{U,X}(g^h)) = 0$ then $\Psi_j^a(g^h) = 0$.*
- *$\Psi_j^a(g^h)) = 0$ if and only if $\mathrm{LM}_{U,X}(g)$ depends on $\{X_1, \ldots, X_{j-1}\}$.*
- *If $\Psi_j^a(g^h)) \neq 0$ then $\Psi_j^a(g^h)) \in \mathbb{Q}[U]$ if and only if $LM_{<_X}(g)$ is a power of X_j.*

Moreover, if G is a reduced Gröbner basis of the ideal I for the monomial ordering $<_{U,X}$ then $\Psi_j^a(G^h)$ is a reduced Gröbner basis of $\Psi_j^a(I^h)$ for the same ordering.

Proof The last assertion is an immediate consequence of the first ones: the proofs that G and $\Psi_j^a(G^h)$ are reduced Gröbner bases are exactly the same, as they involve only the ordering and the leading terms of the polynomials.

The other assertions are immediate consequence of the definition of the Degree Reverse Lexicographical ordering. For any set of variables $Y = Y_1, \ldots, Y_n$ it is defined as: $Y_1^{a_1} \cdots Y_n^{a_n} < Y_1^{b_1} \cdots Y_n^{b_n}$ if and only if $\sum_1^n a_i < \sum_1^n b_i$ or $\sum_1^n a_i = \sum_1^n b_i$ and there is an index j such that $a_j > b_j$ and $\forall i < j, \quad a_i = b_i$. □

With this lemma the end of the proof of theorem 2 is easy:

Since $W_\infty = \pi(\overline{\mathcal{C}}^p \bigcap \mathcal{H}_\infty)$, we want to compute the zeroes of I^h which have a null T-coordinate and at least a non zero X-coordinate. Let $\alpha = (0, u_1, \ldots, u_d, \alpha_{d+1}, \ldots, \alpha_n)$ be such a zero, and suppose that j is the smallest index such that $\alpha_i \neq 0$. Since the polynomials in I^h are homogeneous, α is a zero of I^h if and only if $\alpha' = (0, u_1, \ldots, u_d, \alpha_{j+1}/\alpha_j, \ldots, \alpha_n/\alpha_j)$ is a zero of $\Psi_j^a(I^h)$, which shows that $W_\infty = \bigcup_{j=d+1}^n \overline{\Pi_U(\mathrm{V}(\langle\Psi_j^a(I^h)\rangle))} = \bigcup_{j=d+1}^n \mathrm{V}(\langle\Psi_j^a(I^h) \bigcap \mathbb{Q}(U)\rangle)$. Above lemma shows that $\mathcal{E}_j^\infty = \Psi_j^a(G^h) \bigcap \mathbb{Q}(U)$ is a Gröbner basis of $\langle\Psi_j^a(I^h) \bigcap \mathbb{Q}(U)\rangle$, which proves the second and the third items of the theorem.

The assertion $W_\infty = \Pi$ is equivalent to $\sqrt{\langle\mathcal{E}_0\rangle} = \bigcap_j \sqrt{\langle\mathcal{E}_j^\infty\rangle}$. If $\langle\mathcal{E}_0\rangle$ is prime, then $I \cap \mathbb{Q}[U] =< \mathcal{E}_0 >= \sqrt{< \mathcal{E}_0 >}$ and thus the assertion $W_\infty = \Pi$ is equivalent to $< \mathcal{E}_0 >= \sqrt{< \mathcal{E}_i^\infty >}$ for some i. Since $\langle\mathcal{E}_0\rangle \subset \sqrt{\langle\mathcal{E}_j^\infty\rangle}$, this is again equivalent to $< \mathcal{E}_0 >=< \mathcal{E}_i^\infty >$ for some i; since we have shown that the $\mathcal{E}_j^\infty$ and $\mathcal{E}_0$ are reduced Gröbner bases, the ideals are equals if and only the Gröbner bases are, and the theorem is proved. □

Starting from a Gröbner basis of I for $<_{U,X}$, where $<_U$ and $<_X$are Degree Reverse Lexicographic orderings, the computation of $I_\infty = \cap_{i=0}^n I_i^\infty$ is easy :

Algorithm PROPERNESSDEFECTS

- **Input** : $G_{U,X}$,U,X where $G_{U,X}$ a reduced Gröbner basis w.r.t $<_{U,X}$ where $<_U$ and $<_X$are Degree Reverse Lexicographic orderings;
- **Output** : $I_i^\infty, i = d+1\dots n$ such that
  - I_i^∞ is a Gröbner basis for $<_U$
  - $W_\infty = \cup_{i=0}^{n-d}\mathrm{V}(I_i^\infty)$
- **Begin**
  - **Set** $I_i^\infty = G_{U,X}\cap\mathbb{Q}[U], i = d+1\dots n$
  - **for** $g\in G_{U,X}$
    - **if** $\exists i\in[d+1\dots n]$ and $\exists k\in\mathbb{N}^\star$ such that $\mathrm{LM}_{<_X}(g) = X_i^k$ **then** $I_i^\infty = I_i^\infty\cup\{\mathrm{LC}_{<_X}(g)\}$
  - **return** $(I_i^\infty, d+1 = 1\dots n)$
- **End**

3.3 The core algorithm

In the computation of the minimal discriminant variety, one major problem is the computation of W_c. Characterizing the critical locus of Π_u in restriction to an irreducible variety V of dimension d when knowing any set of generators of $\mathrm{I}(V)$ is easy using the classical Jacobian criterion. Let's define $I_{X,n-\delta} =< I + \mathrm{Jac}_X^{n-\delta}(I) >$ where $\mathrm{Jac}_X^l(I)$ is the ideal generated by all the minors of rank l of the Jacobian matrices of any set of generators of I (does not depend on the chosen set of generators). If $I = \mathrm{I}(V)$ is prime, $V(I_{X,n-\delta})$ is the critical locus of Π_U restricted to V. One can generalize the result to radical and equi-dimensional ideals, but not, for example to non radical or non equi-dimensional ideals. An easy way to avoid problems should be to first compute a decomposition of $\sqrt{I}$ as the intersection of equi-dimensional and radical ideals. For many applications, such a computation is infeasible, and the goal of this section is to find classes of systems for which it can be avoid. We will obviously impose our general algorithm being able to test whether such a computation can be avoid or not.

Next lemma shows that if one replace W_c (resp. W_{sing}) by $\mathrm{V}(I_{X,n-\delta}\cap\mathbb{Q}[U])$ (resp. $\mathrm{V}(I_{U,d-\delta})$) in W_D, one obtain a (non necessarily minimal) discriminant variety even if I is neither radical nor equi-dimensional.

Lemma 7 *Let $I = \cap_{i=1}^{i_1} Q_i \cap_{i=1}^{i_2} Q'_i$ be a minimal primary decomposition of I where $\{Q_i, i = 1\dots i_1\}$ are the primary components Q such that $\dim(Q) = \dim(Q\cap\mathbb{Q}[U])$, and suppose that $W_D = W_\infty\cup W_{\mathrm{sd}}\cup W_{\mathrm{sing}}$ is the minimal discriminant variety of V(I) w.r.t. Π_U.*

If $Q_i = \sqrt{Q_i}, \forall i = 1\dots i_1$, and if $\dim(V((I+\mathrm{Jac}_X^{n-\delta}(I))\cap\mathbb{Q}[U])) < \delta$, then $\dim(W_\infty) < \delta$ and $W_\infty\cup W_{\mathrm{sd}}\cup V((I+\mathrm{Jac}_X^{n-\delta}(I))\cap\mathbb{Q}[U])\cup V((I\cap\mathbb{Q}[U]) + \mathrm{Jac}_U^{d-\delta}(I\cap\mathbb{Q}[U]))$ is a (non necessarily minimal) discriminant variety of V(I) (of dimension $<\delta$) w.r.t. Π_U.

Proof By definition, $W_\infty\subset\cup_{i=1}^{i_2}\mathrm{V}(Q'_i)$. Moreover, if $\dim(Q'_i)\geq\delta$, then $V(Q'_i\cap\mathbb{Q}[U])\subset W_\infty$ and, in particular, $\dim(W_\infty) < \delta$ under our hypothesis. Let p be a critical point of Π_U, $p\in\mathrm{V}(I)$. If p belongs to a unique $\mathrm{V}(Q_i)$, then $\mathrm{rank}(\mathrm{Jac}_X^{n-d}(I)(p)) = \mathrm{rank}(\mathrm{Jac}_X^{n-d}(\sqrt{Q_i})(p)) = \mathrm{rank}(\mathrm{Jac}_X^{n-d}(Q_i)(p))$. In addition, the critical locus of Π_u in restriction to V contains the singular locus of V and thus the intersection of all irreducible components of V. Also, if $i_2 = 0$, then $V(I_{U,\delta}+I)$ is the critical locus of Π_U in restriction to V and $W_c = \mathrm{V}(I_{U,\delta}\cap\mathbb{Q}[U])$. If $i_2\neq 0$, we then have $V((I_{U,\delta}+I)\cap\mathbb{Q}[U])$=$W_c\cup W'$ where $W'\subset\cup_{i=1}^{i_2}\mathrm{V}(Q'_i)$. In the same way, one can see that $V((I\cap\mathbb{Q}[U])+\mathrm{Jac}_U^{d-\delta}(I\cap\mathbb{Q}[U])) = W_{\mathrm{sing}}\cup W''$ where $W''\subset\cup_{i=1}^{i_2}\mathrm{V}(Q'_i)$. For $i\in 1,\dots,i_2$ such that $\dim(Q'_i)\leq\delta$, $\dim(Q'_i\cap\mathbb{Q}[U]) < \delta$, and for $i\in 1,\dots,i_2$ such that $\dim(Q'_i) > \delta$, $\mathrm{V}(Q'_i\cap\mathbb{Q}[U])\subset V((I_{U,\delta}+I)\cap\mathbb{Q}[U])$. Also, if $\dim(V((I_{U,\delta}+I)\cap\mathbb{Q}[U])) < \delta$, $\dim(\cup_{i=1}^{i_2}\mathrm{V}(Q'_i\cap\mathbb{Q}[U])) < \delta$, and thus, since $W\infty\subset\cup_{i=1}^{i_2}\mathrm{V}(Q'_i\cap\mathbb{Q}[U])$, then $\dim(W_\infty) < \delta$. □

Theorem 3 *Using notation of lemma 7, if $I =< f_1,\dots,f_{n-\delta} >$ and if $Q_i = \sqrt{Q_i}, \forall i = 1\dots i_1$, then $W_{\mathrm{sd}} = \emptyset$, $W_\infty\cup W_c = W_\infty\cup V((I+\mathrm{Jac}_X^{n-\delta}(I))\cap\mathbb{Q}[U])$ and $W_\infty\cup W_{\mathrm{sing}} = W_\infty\cup V(I\cap\mathbb{Q}[U]+\mathrm{Jac}_U^{n-\delta}(I\cap\mathbb{Q}[U]))$.*

Proof If $I =< f_1, \ldots, f_{n-\delta} >$, then $\mathrm{V}(I)$ has no irreducible component of dimension $< \delta$ so that, $W_{\mathrm{sd}} = \emptyset$. We are now going to prove that $\cup_{i=1}^{i_2} \mathrm{V}(Q'_i \cap \mathbb{Q}[U]) \subset W_\infty$. If there doesn't exists $i \in 1 \ldots i_2$ such that $\dim(Q'_i) < \delta$, then $\cup_{i=1}^{i_2} \mathrm{V}(Q'_i \cap \mathbb{Q}[U]) \subset W_\infty$. Let $\mathcal{Q} = Q'_i$ with $\dim(Q) < \delta$, $\mathcal{P} = \sqrt{Q}$ and $\mathbb{Q}[U,X]_\mathcal{P}$ be the localization of $\mathbb{Q}[U,X]$ at $\mathcal{P}$. The ideals $I[U,X]$ and $I\mathbb{Q}[U,X]_\mathcal{P}$ are generated by $n-\delta$ elements and so the height of their isolated primary components are $\leq n-\delta$ according to Zariski principal theorem. If $I\mathbb{Q}[U,X]_\mathcal{P}$ has height $< n-\delta$, then $I\mathbb{Q}[U,X]$ has a primary component of height $< n-\delta$ (and thus of dimension $> \delta$ since $\mathbb{Q}[U,X]$ is a Cohen-Macaulay ring) contained in $\mathcal{P}$, which shows that $\mathrm{V}(Q) \subset W_\infty$. Suppose now that $I\mathbb{Q}[U,X]_\mathcal{P}$ has height equal to $n-\delta$. Since $I\mathbb{Q}[U,X]_\mathcal{P}$ is generated by $n-\delta$ elements (the images of the f_i in $\mathbb{Q}[U,X]_\mathcal{P}$), and since $\mathbb{Q}[U,X]_\mathcal{P}$ is a local ring, according to [15], these elements forms a regular sequence. The ring $Q[U,X]_\mathcal{P}$ being Cohen-Macaulay, the ideal $I\mathbb{Q}[U,X]_\mathcal{P}$ is then pure equi-dimensional (have no primary component of height $\neq n-\delta$) and consequently, $I\mathbb{Q}[U,X]$ has no primary component of height $\neq n-\delta$ contained in $\mathcal{P}$, which is impossible since $Q \subset \mathcal{P}$ has dimension $< \delta$ (and so height $> n-\delta$ since $\mathbb{Q}[U,X]$ is Cohen-Macaulay). □

Theorem 3 and lemma 7 shows that one can implement easily and algorithm for computing the minimal discriminant variety without decomposing any ideal in most cases when $\sharp\mathcal{E} = n-\delta$. As written in introduction, most problems coming from applications verify the hypothesis of theorem 3, so that our interest is to compute first $W'_D = W_\infty \cup \mathrm{V}((I + \mathrm{Jac}_X^{n-\delta}(I)) \cap \mathbb{Q}[U]) \cup \mathrm{V}(I \cap \mathbb{Q}[U] + \mathrm{Jac}_U^{n-\delta}(I \cap \mathbb{Q}[U]))$ in any case, and then to test if the conditions of theorem 3 or lemma 7 are fulfilled:

Algorithm CRITICAL

- **Input** : $\mathcal{E}, I, I_\Pi, \delta, U, X$
- **Output** : I_c, I_{sing} and *Property* such that
  - I_c and I_{sing} are reduced Gröbner bases for $<_U$
  - if *Property=Minimal*, then $W_D = W_\infty \cup \mathrm{V}(I_c) \cup \mathrm{V}(I_{\mathrm{sing}}) \cup W_\mathcal{F}$ is the minimal discriminant variety of $\mathcal{C}$ w.r.t. Π_U.
  - if *Property=PartialLargeSD*, then $W_\infty \cup W_{\mathrm{sd}} \cup \mathrm{V}(I_c) \cup \mathrm{V}(I_{\mathrm{sing}}) \cup W_\mathcal{F}$ has dimension $< \delta$ and is a discriminant variety of $\mathcal{C}$ w.r.t. Π_U.
  - if *Property=PartialLargeLD*, then $W_\infty \cup W_{\mathrm{sd}} \cup \mathrm{V}(I_c) \cup \mathrm{V}(I_{\mathrm{sing}}) \cup W_\mathcal{F}$ has dimension δ and is a discriminant variety of $\mathcal{C}$ w.r.t. Π_U (this implies $\delta < d$).
  - if *Property=NeedRadical*, then $W_\infty \cup W_{\mathrm{sd}} \cup \mathrm{V}(I_c) \cup \mathrm{V}(I_{\mathrm{sing}}) \cup W_\mathcal{F}$ is not a discriminant variety of $\mathcal{C}$ w.r.t. Π_U.
- **Begin**
  - **Compute** I_{jac}, the reduced Gröbner basis of $(I \cup \{\mathrm{Jac}_X^{n-\delta}(\mathcal{E})\})$ w.r.t. $<_{U,X}$
  - **Deduce** $I_c = I_{\mathrm{jac}} \cap \mathbb{Q}[U]$
    - ∗ if $\delta < d$ Compute I_{sing}, the reduced Gröbner basis of $(I_\Pi \cup \{\mathrm{Jac}_U^{d-\delta}(I_\Pi)\})$ w.r.t. $<_U$
    - ∗ else set $I_{\mathrm{sing}} = \{1\}$
  - **If** $\dim(I_c) < \delta$ **then**
    - ∗ **if** $n-\delta = \sharp\mathcal{E}$ **then return**(I_c, I_{sing}, *Minimal*)
    - ∗ **else return**(I_c, I_{sing},*PartialLargeSD*)
  - **else**
    - ∗ **if** $(I_c \neq I_\Pi)$ **then return**(I_c, I_{sing},*PartialLargeLD*)
    - ∗ **else return**(I_c, I_{sing},*NeedRadical*)
- **End**

Let $I = \cap_{i=1\ldots j_0} Q_i \cap_{i=1\ldots j_1} Q_i^{(1)} \cap_{i=1\ldots j_2} Q_i^{(2)}$ be a minimal primary decomposition of I where $(Q_i^{(1)})_{i=1\ldots j_1}$ are the primary but not prime components such that $\dim(Q_i^{(1)}) = \delta = \dim(Q_i^{(1)} \cap \mathbb{Q}[U])$, and $(Q_i^{(2)})_{i=1\ldots j_2}$ those such that $\dim(Q_i) = \dim(Q_i \cap \mathbb{Q}[U]) < \delta$ and such that there does not exists any Q_j with $\dim(Q_j) = \dim(Q_j \cap \mathbb{Q}[U]) = \delta$ and $V(Q_i) \subset V(Q_j)$. Then, the minimal discriminant variety of $I' = \cap Q_i \cap \sqrt{Q_i^{(1)}}$ is the minimal discriminant variety of I. One can then analyze the output of the algorithm CRITICAL:

Remark 3 If algorithm CRITICAL returns the message :

- *PartialLargeLD* or *NeedRadical*, then $((W_\infty \cup W_{\text{sd}} \cup \text{V}(I_c) \cup \text{V}(I_{\text{sing}}) \cup W_{\mathcal{F}}) \setminus W_D) \subset (\cup_{i=1\ldots j_1} V(Q_i^{(1)}) \cup_{i=1\ldots j_2} V(Q_i^{(2)}))$
- *PartialLargeSD*, then $((W_\infty \cup W_{\text{sd}} \cup \text{V}(I_c) \cup \text{V}(I_{\text{sing}}) \cup W_{\mathcal{F}}) \setminus W_D) \subset \cup_{i=1\ldots j_2} V(Q_i^{(2)})$

In particular, if $j_1 = j_2 = 0$, $W_\infty \cup W_{\text{sd}} \cup \text{V}(I_c) \cup \text{V}(I_{\text{sing}}) \cup W_{\mathcal{F}}$ is the minimal discriminant variety of $\mathcal{C}$ with respect to Π_U.

Let us now combine the algorithms PREPROCESSING, PROPERNESSDEFECTS and CRITICAL to propose an algorithm which produces an acceptable output for almost all applications (see section 3.1):

Algorithm CORE

- **Input** : $\mathcal{E}, \mathcal{F}, U, X$
- **Output** : $I, I_\Pi, \delta, k, I_{D,1}, \ldots, I_{D,k}$ and *Property* such that
  - I is a reduced Gröbner basis for $<_{U,X}$ where $<_U$ and $<_X$ are Degree Reverse Lexicographic orderings
  - $\mathcal{C} \cap \Pi_U^{-1}(\Pi \setminus W_D) = \text{V}(I) \cap \Pi_U^{-1}(\Pi \setminus W_D)$
  - $k \in \mathbb{N}^+$
  - $\delta = \dim(I_\Pi)$
  - $(I_{D,i})_{i=1\ldots k}$ and I_Π are Gröbner bases for $<_U$
  - if *Property=Minimal*, then $\cup_{i=1}^k \text{V}(< I_{D,i} >)$ is the minimal discriminant variety of $\mathcal{C}$ w.r.t. Π_U;
  - if *Property=PartialLargeSD*, then $W_{\text{sd}} \cup_{i=1}^k \text{V}(< I_{D,i} >)$ has dimension $< \delta$ and is a discriminant variety of $\mathcal{C}$ w.r.t. Π_U;
  - if *Property=PartialLargeLD*, then $W_{\text{sd}} \cup_{i=1}^k \text{V}(< I_{D,i} >)$ is a discriminant variety $\mathcal{C}$ w.r.t. Π_U (and $\delta < d$).
  - if *Property=NeedRadical*, then $W_{\text{sd}} \cup W_c \cup W_{\text{sing}} \cup_{i=1}^k \text{V}(< I_{D,i} >)$ is the minimal discriminant variety of $\mathcal{C}$ w.r.t. Π_U.
- **Begin**
  - $\delta, I, I_\Pi, I_{\mathcal{F}}$=PREPROCESSING($\mathcal{E}, \mathcal{F}, U, X$)
  - **if** ($I_\Pi = I_{\mathcal{F}}$) **then return**(I,$I_\diamond$,δ,1,I_Π,*Minimal*)
  - $(I_i^\infty)_{i=1\ldots n-d}$=PROPERNESSDEFECTS($I$,$U$,$X$)
  - **if** ($I_i^\infty \subset < I_\Pi >, i = 1\ldots n$) **then return**($I$,$I_\diamond$,$\delta$,1,$I_\Pi$,*Minimal*)
  - I_c, I_{sing},*Property*=CRITICAL($\mathcal{E}, I, I_\Pi, \delta, U, X$)
  - *if Property=NeedRadical*, **then return**($I, I_\Pi, \delta, n-d+1, I_{\mathcal{F}}, (I_i^\infty)_{i=1\ldots n-d}$,*Property*)
  - **else return**($I, I_\Pi, \delta, n-d+3, I_{\mathcal{F}}, (I_i^\infty)_{i=1\ldots n-d}$,,$I_c, I_{\text{sing}}$,*Property*)
- **End**

3.4 Discriminant varieties and triangular sets

We have shown that Gröbner bases allow to compute minimal discriminant varieties for a large range of problems. The exceptions are listed in remark 3 and suggest to compute some lazy decomposition of the input system to be able to provide the minimal discriminant variety for each kind of system. One should modify some existing algorithms that compute primary or equi-dimensional decompositions but we choose to use decomposition into triangular sets for two main reasons :

- they seem to be one of the most efficient approach to decompose an ideal into radical and equi-dimensional components;
- we will need to apply the algorithm CRITICAL on each component which is more easy in the case of triangular systems since the number of polynomials is exactly equal to the co-dimension;

Let's recall some well known definitions (see [23] for details).

A triangular set with respect to the ordering s.t. $Y_1 < \ldots < Y_n$ is a set of polynomials with the following shape:

$$\begin{cases} t_1(Y_1) \\ t_2(Y_1, Y_2) \\ \vdots \\ t_n(Y_1, \ldots, Y_n) \end{cases}$$

Some of the t_i may be identically zero.

Notation 3 *For $p \in \mathbb{Q}[Y_1, \ldots, Y_n] \setminus \mathbb{Q}$, we denote by mvar$(p)$ (and we call main variable of p) the greatest variable appearing in p w.r.t. the chosen ordering.*

- *h_i the leading coefficient of t_i (when $t_i \neq 0$ it is seen as a univariate polynomial in its main variable), and $h = \Pi_{i=1,t_i\neq 0}^n h_i$.*
- *$s_i = \frac{\partial t_i}{\partial Y_i}$ the separant of t_i (when $t_i \neq 0$), and $s = \Pi_{i=1,t_i\neq 0}^n s_i$.*
- *sat$(T) = \langle T \rangle : h^\infty = \{p \in \mathbb{Q}[Y_1, \ldots, Y_n] \mid \exists m \in \mathbb{N}, \quad h^m p \in \langle T \rangle\}$; the elements of the set V$(T) \setminus$ V(h) are called regular zeroes of T and the variety $\overline{\mathrm{V}(T) \setminus \mathrm{V}(h)} = \mathrm{V}(sat(T))$ (elementary property of localization) is called the variety of T.*
- *the normalization of a polynomial with respect to a triangular set is the result of the following computation: normalize(p, T) == for i from n down to 1 do if $t_i \neq 0$ then p :=resultant(p, i, Y_i) fi od; return p.*
- *the dimension dim(T) of a triangular set T is the number of t_i which are null; the ideal $sat(T)$ and the corresponding variety are equi-dimensional of dimension dim(T).*

Definition 4 *A triangular set $T = (t_1, \ldots, t_n) \subset \mathbb{Q}[Y_1, \ldots, Y_n]$ is said to be regular (resp. separable) if, for each $t_i \neq 0$, the normalization of its initial h_i (resp. of its separant s_i) is a non zero polynomial.*

One can always decompose a variety as the union of the varieties of regular and separable triangular sets ([1],...):

$$V_C = \bigcup_i \mathrm{V}(\mathrm{sat}(T_i)). \tag{1}$$

Such a decomposition is said not redundant, if for any T_i, none of the irreducible components of the variety of T_i is contained in the variety of another T_j. Note that, to test this and to compute an not redundant decomposition, the known algorithms may need to compute a Gröbner basis of sat(T_j), when dim$(T_j) >$ dim(T_i).x

The following result makes complete the well known properties of the triangular sets that we need.

Proposition 2 *Given a regular triangular set $T = (t_1, \ldots, t_n)$, then $(t_1, \ldots, t_i)$ is a regular triangular set, the variety of which is the Zariski closure of the projection of the variety of T on the affine space corresponding to the variables $Y_1, \ldots, Y_i$.*

We are now ready to use triangular sets in our problem.

Theorem 4 *Let us consider a decomposition (1) of our variety in regular separable triangular systems. Let us denote* $V_U(T_i) = \overline{\Pi_U(V(sat(T_i)))}$*, the Zariski closure of the projection of the variety of* T_i *on the U-space. Then*

- $W_{sd} \subset \bigcup_{dim(T_i)<\delta} V_U(T_i)$. *If the decomposition is not redundant, then this inclusion is an equality.*
- $W_\infty = \bigcup W_\infty(sat(T_i))$. *If* $dim(T_i) < \delta$ *then* $W_\infty(sat(T_i)) \subset W_{sd} \cup \bigcup_{dim(T_j)\geq\delta} W_\infty(sat(T_j))$. *If* $dim(T_i) > \delta$ *then* $W_\infty(sat(T_i)) \subset W_c$. *Thus* W_∞ *has to be computed only for components of dimension* δ.
- W_c *is the union of* $\bigcup_{dim(T_i)=\delta} W_c(sat(T_i))$, *of* $\bigcup_{dim(T_i)=\delta,\ dim(T_j)=\delta} \overline{\Pi_U(\mathcal{V}(sat(T_i)+sat(T_j))}$, *of* $\bigcup_{dim(T_i)>\delta} V_U(T_i)$ *and possibly of some components included in* W_{sd}.
- $W_{f_i} = \bigcup W_{f_i}(sat(T_i))$. *If* $dim(T_i) < \delta$ *then* $W_{f_i}(sat(T_i)) \subset W_{sd} \cup \bigcup_{dim(T_j)\geq\delta} W_{f_i}(sat(T_j))$. *If* $dim(T_i) > \delta$ *then* $W_{f_i}(sat(T_i)) \subset W_c$. *Thus* W_{f_i} *has to be computed only for components of dimension* δ.
- *If the variable ordering is such that* $u_1, \ldots, u_d$ *are the smallest variables, then* $V_U(T_i)$ *is the variety of the (regular separable) triangular set* $T_i \cap \mathbb{Q}[U]$*, and its dimension is the number of null polynomials in this intersection.*

Proof The second item is a simple translation of the definition of W_{sd}, using the equidimensionality of the V(sat(T_i)).

The definition of W_∞ is clearly invariant when a variety is decomposed as a union of varieties. Moreover the W_∞ of a variety is contained in the Zariski closure of the projection of the variety, and if a variety is included in another one, the same is true for their W_∞. This completes the proof of the third item, by using the fact that the projection of a variety of dimension higher than δ is included in its W_c, which will be shown below.

For W_{f_i}, the proof is exactly the same.

A union of varieties $V_1 \cup V_2$ may be defined by the product of their ideal of definition. The rule of derivation of a product, shows that a row of the Jacobian matrix of $V_1 \cup V_2$ at a point which is on V_1 is proportional to a row of the Jacobian matrix of V_1. Thus, if the point is not on V_2 the rank of the Jacobian matrices of V_1 and $V_1 \cup V_2$ are the same. If the point is on both V_1 and V_2 the Jacobian matrix is null. Finally, if the projection reduces the dimension of a variety, it is an exercise to show that the rank of the Jacobian matrix we are considering is less than $n - d$.

As the last item is simply a translation of Proposition 2, this completes the proof. □

This theorem provides much more information when the variables $u_1, \ldots u_d$ are the smallest ones. However, it has been stated for any ordering, because, it may be the case that the decomposition in triangular sets is much easier for some ordering of the variables. In such a case, this allows to split the computation in several (a priori) easier ones, or, if there is only one component, to avoid the computation of W_{sd} which is empty. The following system, which comes from celestial mechanics and is the computationally hardest we have ever treated, shows that any ordering on the variables may be useful.

Example 1 Consider the equations $(b-d)^2 - 2(b+d) + 1 + f = 0, m(B-1) - (D-F)(d-b+1) = 0, n(D-1) - (B-F)(b-d+1) = 0, b^3B^2 = 1, d^3D^2 = 1, f^3F^2 = 1$, where the parameters ($U$-variables) are m and n and the unknowns (X variables) are the six other ones. The inequalities are simply that any variable should be positive. It is almost immediate that this set of polynomials is a regular separable triangular T set of dimension 2 for the ordering $m > n > B > D > F > f > b > d$. It is also immediate that the set is irreducible, which means that each polynomial is irreducible modulo the polynomials of lower main variable (this is true for the polynomials in m and n, which are linear; the other ones are or may be rewritten as irreducible polynomials with coefficients in $\mathbb{Q}[b,d]$). This implies that sat(T) is a prime ideal. The complex solutions of the system where some initial is null (i.e. $B = 1$ or $D = 1$) may be easily decomposed in components of dimension 1. As the system is generated by 6 equations, its variety may not have components of dimension ≤ 1. This shows that the ideal generated by the equations is prime, equal to sat(T), that W_{sd} is empty and that any not redundant decomposition in triangular sets, for any ordering, reduces to a single triangular set.

For completing the computation of a discriminant variety using triangular sets, it remains to compute W_c, W_∞ and W_{f_i} for the variety of a triangular set of dimension δ and also the projection of the intersection of the varieties of two triangular sets.

Theorem 5 *Let $T = (t_1, \ldots, t_n)$ be a regular separable triangular set of dimension δ for an ordering such that the variables in U are the lower ones (i.e. $\{t_1, \ldots, t_d\} \subset \mathbb{Q}[U]$). Let h_i and s_i be the initial and the separant of t_i, and $h_U = \prod_{i>d, t_i \neq 0}$ normalize(h_i, T), $s_U = \prod_{i>d, t_i \neq 0}$ normalize(s_i, T).*

The variety W_∞ associated to the variety of T is included in the variety of h_U, and W_c is included in the variety of s_U. Even when $d = \delta$ these inclusions may be strict.

If f_i is a polynomial such that $f_U :=$ normalize$(f_i, T) \neq 0$, then W_{f_i} is included in the variety of f_U. If $f_U = 0$, some resultant involved in the computation of normalize is null, and the corresponding t_i has a non trivial factor. By standard algorithms, this allows to decompose T in triangular sets such that either their f_U is not null, either their variety is included in the variety of f_i. Thus, W_{f_i} is included in the variety of the product of the non zero f_U.

Proof Let us consider a sequence of points of the variety of T tending to the infinity. Let x_i be the lowest variable tending to the infinity. Necessarily, h_i tends to 0. As normalize(h_i, T) is in the ideal generated by h_i and the t_i (property of the resultants), it tends also to 0. This shows that the limit lies in the variety of h_U.

As the t_i are in the ideal of the variety of T, the Jacobian of $(t_{i+1}, \ldots, t_n)$ (with respect to the x_i) vanishes on the critical points of the projection. This Jacobian is $\prod_{i>d} s_i$. As above, this implies that normalize$(\prod_{i>d} s_i, T)$ vanish also at these points. This polynomial is s_U, because the resultants and therefore the function normalize are compatible with products.

The proof of the first assertion concerning W_{F_i} is similar. the others are immediate.

The fact that the inclusions may be strict may be seen on the following simple example: let us consider the polynomials $(x^2 + y^2 - 1, ux - vy)$. The decomposition in triangular sets for $v < u < y < x$ is simply $T = (0, 0, (u^2 + v^2)y^2 + u^2, ux - vy)$. Thus $h_U = u(u^2 + v^2)$ and $s_U = u^2(u^2 + v^2)$, while it is easy to verify that W_∞ and W_c are both defined by $u^2 + v^2 = 0$. $\square$

Given two triangular sets T and T', the decomposition in triangular sets of the intersection of their varieties may be provided by the algorithms that compute decompositions into triangular sets; it may also be done by computing the ideal sat(T) + sat(T') and compute the projection of its variety through Gröbner bases, which is usually easy.

3.5 A general algorithm

In this section, we propose a general algorithm for the computation of discriminant varieties. This version first runs the algorithm CORE. If it returns with a message different from *Minimal* we then compute a decomposition of $\overline{\mathcal{C}}$ as the union of the zero sets of equi-dimensional and radical ideals through regular and separable triangular sets as described in the above section. In practice, we use the implementation from the RAGLIB library [18] : it provides in the meantime the triangular sets (T_i) as well as their saturated ideals ($\mathrm{sat}(T_i)$). Note that many variants could be derive from this general algorithm depending on the user's request.

Algorithm DISVAR

- **Input:** $\mathcal{E}, \mathcal{F}, U, X$, *NeedMinimal =true* if and only if the minimal discriminant variety is requested, *NeedSmallDim =true* if and only if W_{sd} is requested.
- **Ouput:** $I_{D,1}, \ldots, I_{D,k}$ and *Property* such that
  - $I_{D,i}, i = 1 \ldots k$are Gröbner bases for $<_{U,X}$
  - if *NeedMinimal=true,* $\cup_{i=1\ldots k} \mathrm{V}(I_{D,i})$ is the minimal discriminant variety of $\mathcal{C}$ w.r.t. Π_U.
  - other else
    - if *NeedSmallDim=true,* $\cup_{i=1\ldots k} \mathrm{V}(I_{D,i})$ is a discriminant variety of $\mathcal{C}$ w.r.t. Π_U;
    - other else $W_{\mathrm{sd}} \cup_{i=1\ldots k} \mathrm{V}(I_{D,i})$ is a discriminant variety of $\mathcal{C}$ w.r.t. Π_U.

Begin

- $I, I_\Pi, \delta, k, (I_{D,i})_{i=1\ldots k}$, *Property*=CORE$(\mathcal{E}, U, X)$
- $I_D = \{I_{D,i}, i = 1 \ldots n - d + 1\}$ *(at this step : V$(I_D) = W_\infty \cup W_{\mathcal{F}}$)*

- **if** (*Property* $\neq$ *Minimal*) and (*NeedMinimal* or (*NeedSmallDim* and *NeedRadical*)) **then**
  - $(\mathcal{T}_i, \mathrm{sat}(\mathcal{T}_i))_{i=1\ldots m} = \mathrm{DECOMPOSE}(\mathcal{E})$
  - **for** $i = 1 \ldots m$ **do**
    * **if** $((\dim(\mathcal{T}_i) < \delta)$ and $(\dim(\mathcal{T}_i \cap \mathbb{Q}[U]) = \dim(\mathcal{E}_i)))$ **then** $I_D = I_D \cup \{\{\mathcal{T}_\rangle \cap \mathbb{Q}[U]\}\}$
    * **if** (*NeedMinimal* or *NeedRadical*) and $(\dim(\mathcal{T}_i) = \delta = \dim(\mathcal{T}_i \cap \mathbb{Q}[U]))$ **then**
      · $I_D = I_D \cup \{\ \mathrm{CRITICAL}(\mathcal{T}_\rangle, \mathrm{sat}(\mathcal{T}_i), \mathcal{T}_\rangle \cap \mathbb{Q}[U], \delta, U, X)\}$
      · **for** $b = i \ldots m$ **do** $I_D = I_D \cup \{\{(\mathrm{sat}(\mathcal{T}_i) + \mathrm{sat}(\mathcal{T}_b)) \cap \mathbb{Q}[U]\}\}$
    * **else** $I_D = I_D \cup \{I_{D,i}, i = n - d \ldots k\}$
- **else** $I_D = I_D \cup \{I_{D,i}, i = n - d \ldots k\}$
- **return**(I_D)

End

4 Using discriminant varieties in practice

Let us suppose that W_D is a discriminant variety of $\mathcal{C}$ w.r.t. Π_U, let denote by $\mathcal{U}_1, \ldots \mathcal{U}_k$ the sub-manifolds of $\Pi \setminus W_D$ as introduced in definition 1 and set $\mathcal{V} = \mathcal{C} \bigcap \left(\bigcup_{i=1}^{k} \Pi_U^{-1}(\mathcal{U}_i) \right)$.

If $u_1, \ldots, u_k$ are sample points such that $u_i \in \mathcal{U}_i$ then $\bigcup_{i=1}^{k} \Pi_U^{-1}(u_i)$ intersects each connected component of $\mathcal{V}$ (this is a direct consequence of definition 1) and define exactly one point on each connected component of $\mathcal{V}$. In the real case, by removing the points of $\Pi_U^{-1}(u_i)$ that do not verify the inequations $f > 0, f \in \mathcal{F}$ one then compute exactly one point on each semi-algebraically connected component of $\mathcal{V} \bigcap \mathcal{S}$. Thus, by computing one point on each $\mathcal{U}_\rangle$, one can, test if there are real points in $\mathcal{V}$ or $\mathcal{V} \bigcap \mathcal{S}$, compute the maximal or minimal number of real/complex solutions of $\mathcal{S}$ of $\mathcal{C}$ for parameters' values that do not belong to W_D or compute (exactly) one point on each connected component of $\mathcal{V}$ or $\mathcal{V} \bigcap \mathcal{S}$.

Obtaining test points such as $u_1, \ldots, u_k$ remains to compute one point on each connected component of $\Pi \setminus W_D$. This can theoretically be solved with a good complexity by the algorithms described in [4]. In practice, the end-user often wants to compute the number of real roots of the system w.r.t. parameter's values. Computing at least one point on each $\mathcal{U}_i$ not sufficient in this case since a suitable description of the $\mathcal{U}_i$ is then required.

Basically, the CAD algorithm computes a cylindrical decomposition of the ambient space in cells such that the polynomials of a given set have a constant sign on each cell. Precisely :

Definition 5 *A cylindrical algebraic decomposition of $\mathbb{R}^d$ is a sequence $C_1, ..., C_d$, where, for $1 \leq k \geq d$, C_k is a finite partition of $\mathbb{R}^k$ into semi-algebraic subsets (which are called cells), satisfying the following properties:*

- *Each cell $C \in C_1$ is either a point, or an open interval.*
- *For every $k, 1 \leq k < d$, and for every $C \in C_k$, there are finitely many continuous semi-algebraic functions $\xi_{C,1} < ... < \xi_{C,l_C}$ $C : C \longrightarrow \mathbb{R}$ and the cylinder $C \times \mathbb{R} \subset \mathbb{R}^{k+1}$ is the disjoint union of cells of C^{k+1} which are:*
  - *either the graph of one of the functions $\xi_{C,j}$, for $j = 1, ..., l_C$:*
  $$A_{C,j} = \{(x', x_{k+1}) \in C \times \mathbb{R} \quad ; \quad x_{k+1} = \xi_{C,j}(x')\};$$
  - *or a band of the cylinder bounded from below and from above by the graphs of functions $\xi_{C,j}$ and $\xi_{C,j+1}$, for $j = 0, \ldots, l_C$, where we take $\xi_{C,0} = -\infty$ and $\xi_{i,l_C+1} = +\infty$:*
  $$B_{C,j} = \{(x', x_{k+1}) \in C \times \mathbb{R} \quad ; \quad \xi_{C,j}(x') < x_{k+1} < \xi_{C,j+1}(x')\}.$$

A CAD adapted to any set $\{P_1, \ldots, P_s\}$ of polynomials of $\mathbb{R}[U_1, \ldots, U_d]$ is a CAD such that each cell C is $(P_1, \ldots, P_s)$-invariant, which means that the P_i have a constant sign in each cell.

It is always possible to compute a CAD of $\mathbb{R}^d$ adapted to a given set of polynomials ([7]). A straightforward method would then consists in computing a CAD of $\mathbb{C}^d$, adapted to the polynomials whose zeroes sets are Π and W_D (which are explicitly known after the computation of a discriminant variety) : the sub-manifolds $\mathcal{U}_1, \ldots, \mathcal{U}_k$, will then be described as unions of cells included in Π and of dimension (δ), which are easy to detect in practice. There exists some well suited versions of the CAD algorithm that can compute a so called Partial CAD of $\mathbb{R}^d$ which can directly compute the cells in which we are interested [6]. In our case, the most costly operations (computations with real algebraic numbers) can then be avoid.

At this step, one have a partition of Π constituted by W_D and a collection of cells of dimension δ. Note that compared with a partial CAD adapted to $\mathcal{E} \cup \mathcal{F}$, we have replaced the $n - \delta$ projection step by the computation of W_D.

5 Some Applications

In this section, we revisit some applications already solved by ad-hoc computations. The goal is to illustrate, on non-trivial and practical examples, how to solve efficiently some problems dealing with parametric systems using our algorithms as black-boxes. We chose problems known to be difficult but so that the solutions we propose can easily be reproduced by the reader.

5.1 Cuspidal serial manipulators

We revisit here a ad-hoc computation done in [8]. The goal was to compute a classification of 3-revolute-jointed manipulators based on the cuspidal behavior. This ability to change posture without meeting a singularity is equivalent to the existence of a point in the workspace, such that a polynomial of degree four depending on the parameters of the manipulator and on the Cartesian coordinates of the effector has a triple root.

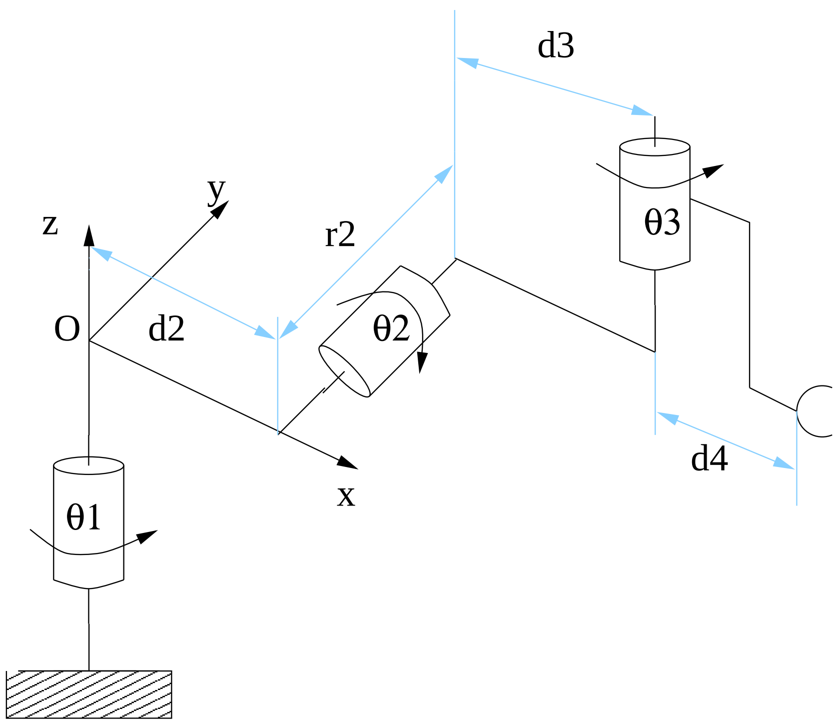

Figure 1: A Manipulator

The system that characterizes the cuspidal robots depends on 3 parameters d_4, d_3 and r_2 which are the design parameters (supposed to be positive). It is given by:

$$P(t) = at^4 + bt^3 + ct^2 + dt + e = 0, \frac{\partial P}{\partial t} = 0, \frac{\partial^2 P}{\partial t^2} = 0, d_4 > 0, d_3 > 0, r_2 > 0$$

with:

$$\left\{\begin{array}{lcl} a & = & m_5 - m_2 + m_0 \\ b & = & -2m_3 + 2m_1 \\ c & = & -2m_5 + 4m_4 + 2m_0 \\ d & = & 2m_3 + 2m_1 \\ e & = & m_5 + m_2 + m_0 \\ m_0 & = & -r^2 + r_2^2 + \frac{(R+1-L)^2}{4} \\ m_1 & = & 2r_2d_4 + (L-R-1)d_4r_2 \\ m_2 & = & (L-R-1)d_4d_3 \\ m_3 & = & 2r_2d_3d_4^2 \\ m_4 & = & d_4^2(r_2^2+1) \\ m_5 & = & d_4^2d_3^2 \\ r^2 & = & x^2+y^2 \\ R & = & r^2+z^2 \\ L & = & d_4^2+d_3^2+r_2^2 \end{array}\right.$$

In [8], the authors used a particular change of variables and a ad-hoc method based on decompositions into triangular sets to compute a discriminant variety. The final decomposition of the parameter's space was obtained using a partial Cylindrical Algebraic Decomposition [6].

Let us show how the method proposed in this article allows to solve automatically the problem. The example is interesting since the equi-dimensional decomposition of $\mathcal{E}$ is very difficult to compute in practice.

We take $\mathcal{E} = \{P, \frac{\partial P}{\partial t}, \frac{\partial^2 P}{\partial t^2}\}$, $\mathcal{F} = \{d_4, d_3, r_2\}$, $U = [d_4, d_3, r_2]$ and $X = [t, z, r]$. The system has dimension 4 but the only component of dimension 4 is embedded in $\mathrm{V}(d_4) \subset W_{\mathcal{F}}$ to that the algorithm PREPROCESSING do not perform any localization and its output is :

- $\delta = 3$;
- I is the Gröbner basis of $\mathcal{E}$ for $<_{U,X}$;
- $I_\Pi = \{\}$;
- $I_{\mathcal{F}} = \{d4\} \cup \{d3\} \cup \{r2\}$;

As in most situations, W_∞ is easy to compute. On this example, the result the algorithm PROPERNESSDEFECTS returns :

- $I_4^\infty = \{1\}, I_5^\infty = \{r2d4 - d3r2 + r2^3d4\}, I_6^\infty = \{1\}$

Since $< \mathcal{E} > + \mathrm{Jac}_X^{n-d}(\mathcal{E})$ has dimension $< d$ and since the system has 3 equations and depends on 3 parameters, then $W_D = \cup_{i=4\ldots6}\mathrm{V}(I_i^\infty) \cup \mathrm{V}((< \mathcal{E} > + \mathrm{Jac}_X^{n-d}(\mathcal{E})) \cap \mathbb{Q}[U])$. The output of the algorithm CRITICAL is :

- *Property=Minimal*
- $I_{\text{sing}} = \{\}$
- $I_{\text{crit}} =$

```
{-d4^2+r2^2+d3^2,
  d4^2*r2^6-d4^4*r2^4+2*d4^2*r2^4+
  3*d4^2*d3^2*r2^4-2*d4^4*r2^2+d4^2*r2^2-
  2*d4^4*d3^2*r2^2+3*d4^2*d3^4*r2^2-d3^2*r2^2-
  d4^4*d3^4+d4^2*d3^2+d4^2*d3^6-2*d4^2*d3^4-d4^4+2*d4^4*d3^2,
```

```
r2^8+2*d3^2*r2^6+2*r2^6-2*d4^2*r2^6+d4^4*r2^4-
4*d4^2*r2^4-2*d3^2*r2^4-2*d4^2*d3^2*r2^4+r2^4+d3^4*r2^4-
2*d4^2*r2^2+2*d4^4*r2^2+2*d4^2*d3^2*r2^2+d4^4,
d3^2*r2^2-d4^2+2*d4^2*d3+d3^2-d4^2*d3^2-2*d3^3+d3^4,
d3^2*r2^2-d4^2-2*d4^2*d3+d3^2-d4^2*d3^2+2*d3^3+d3^4}
```

Also, even if we apply directly the algorithm DISVAR with the option *NeedMinimal=true*, no decomposition will be computed and at most 2 Gröbner bases computations for a degree - block ordering $<_{U,X}$ are needed (I and I_{crit}).

Removing the polynomials that have no real roots, our algorithm gives exactly the same result as the one obtained in [8]. As in [8], one can terminate the computations easily using the partial CAD and some tools for computing the real roots of zero-dimensional systems. The projection of the discriminant variety on the subspace (d_3, r_2) (obtained after the first partial CAD projection step):

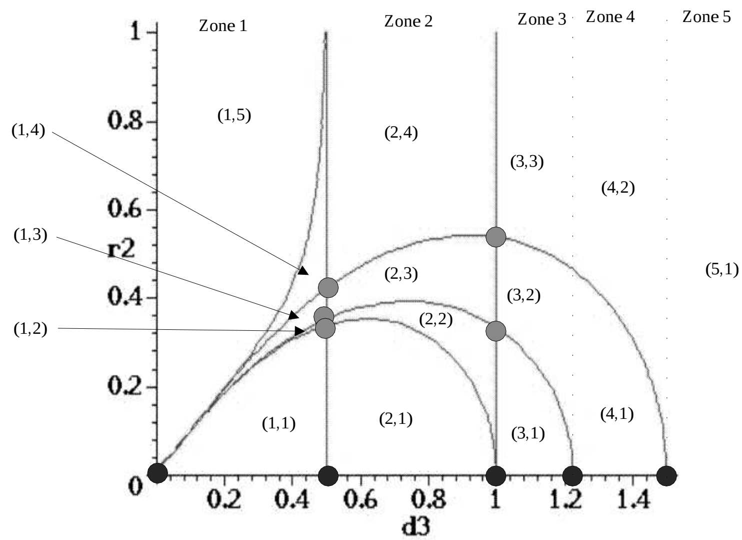

Figure 2: The partition of the parameters'space (d_3, r_2)

Here are the results for each sample point:

$(d_3, r_2) \setminus d_4$	1	2	3	4	5	6	7
(1,1)	0	0	4	4	2	0	0
(1,2)	0	4	4	4	2	0	0
(1,3)	0	4	4	4	2	0	0
(1,4)	0	4	4	2	2	0	0
(1,5)	0	4	4	2	0	0	0
(2,1)	0	0	4	4	2	2	0
(2,2)	0	4	4	4	2	2	0
(2,3)	0	4	4	4	2	2	0
(2,4)	0	4	4	2	2	2	0
(3,1)	0	4	4	4	2	2	4
(3,2)	0	4	4	4	2	2	4
(3,3)	0	4	4	2	2	2	4
(4,1)	0	4	4	4	2	2	4
(4,2)	0	4	4	2	2	2	4
(5,1)	0	4	4	2	2	2	4

In each line (i, j) of this table you can read the number of cusp points appearing in a cross section of the workspaces of the seven test robots above the cell (i, j) corresponding to the seven distinct test values of d_4 obtained .

We can assume that the problem is completely solved, even if no precise information is known for parameter's values that belongs to the discriminant variety: it will anyway be impossible to construct, in practice, robots whose parameters belong to a strict closed subset of the parameter's space.

5.2 Equi-Cevaline points on triangles

The problem proposed in [27] has been already solved by the authors, partially by "hand". In their article, they asked for a general solver able to produce the same kind of results. In [13], the author gave a partial answer and list the main specifications of such a solver. Within this section we show that our algorithm fits these requirements.

The goal is to study the points of $\mathbb{R}^3$ where three lines passing through one point P and one vertex of a triangle intersects the triangle in three segments of same length. In the following system, the parameters a, b, c represent the lengths of the sides of a triangle ABC, l is the common length of the intersections and x, y, z are the homogeneous barycentric coordinates of P:

$$\begin{aligned}
&p_1 := (c^2 - l^2) * y^2 + (b^2 - l^2) * z^2 + (b^2 + c^2 - a^2 - 2 * l^2) * y * z = 0;\\
&p_2 := (a^2 - l^2) * z^2 + (c^2 - l^2) * x^2 + (c^2 + a^2 - b^2 - 2 * l^2) * z * x = 0;\\
&p_3 := (b^2 - l^2) * x^2 + (a^2 - l^2) * y^2 + (a^2 + b^2 - c^2 - 2 * l^2) * x * y = 0;\\
&f_1 := x + y + z = 1;
\end{aligned}$$

After substituting x by $1-y-z$, and according to the notations introduced in the present paper, we take $\mathcal{E} = \{p_1, p_2, p_3\}$, $\mathcal{F} = f_1$, $X = [y, z, l]$ and $U = [a, b, c]$. In [13], the author defines in fact a discriminant variety constituted by conditions of degeneracy (degenerated triangles) and 10 polynomials. As for the previous application, there is no need to localize by f_1, W_{sing} and W_{sd}. Also, it only remains to compute W_∞ and W_c. Our algorithm computes easily $W_\infty = \mathrm{V}(abc(a + c - b)(a + c + b)(b + a - c)(a - b - c))$, which corresponds exactly to the degenerated situations (degenerated triangles) listed by the authors in [27] and [13]. W_c is then the union of the zero sets of the following polynomials:

```
[c+b,
c-b,
1/5*c^4-2/5*a^2*c^2+a^4-2/5*b^2*c^2-2/5*a^2*b^2+1/5*b^4,
-5/3*c^4+2/3*a^2*c^2+a^4+2/3*b^2*c^2-2*a^2*b^2+b^4,
a+c,
c^4-a^2*c^2+a^4-b^2*c^2-a^2*b^2+b^4,
5*c^4-2*a^2*c^2+a^4-2*b^2*c^2-2*a^2*b^2+b^4,
a-b,
a+b,
a-c,
c^4-2*a^2*c^2+a^4-2*b^2*c^2-2*a^2*b^2+5*b^4,
c^4-2*a^2*c^2+a^4+2/3*b^2*c^2+2/3*a^2*b^2-5/3*b^4,
-3/5*c^4-2/5*a^2*c^2+a^4+6/5*b^2*c^2-2/5*a^2*b^2-3/5*b^4]
```

By removing the polynomials that have no real roots in the first quadrant, we obtain the same 10 polynomials as in [13]. In other words, our algorithm is an automatic method for solving the problem.

6 Conclusion and perspectives

In this article, we proposed some tools for computing the discriminant variety of a basic constructible $\mathcal{C}$ (w.r.t. a given projection) set and for describing the sub-manifolds of its complementary in the Zariski closure of the image of $\mathcal{C}$. We shown that this object is optimal and easy to compute in most cases. We also demonstrate its efficiency in terms of computation times (cuspidal manipulators) but also in terms of quality of the output (Equi-Cevaline points on triangles).

The perspectives are of two kind: complexity and generalization to more general problems. For the first item, we already know, since the object is optimal and so already computed in several algorithms, that the degree of the discriminant variety is singly exponential in the number of variables at least in the case where $\Pi = \mathbb{C}^d$ (see [12] for example). Computing a precise bound would give precious informations on the complexity of solving parametric systems. We also know that, in the case where $\Pi = \mathbb{C}^d$, the running time of our algorithm is singly exponential in the number of variables if we use the Gröbner engine proposed in [12]. A challenge would be to prove we would get a better or at least more precise bound by using [10] as Gröbner engine.

Our method may be easily adapted for solving more general problems. We can, for example, fully describe $\mathcal{C}$ w.r.t. the parameters by applying our algorithm on $\mathcal{E} \cup \mathcal{F} \cup W_D$ or $\mathcal{E} \cup \mathcal{F} \cup (W_D \setminus W_\infty)$ (and so on, recursively). One also can extend the method in order to solve general positive dimensional systems as follows. Suppose that $\mathcal{E}$ and $\mathcal{F}$ are subsets of $\mathbb{Q}[Y_1, \ldots, Y_n]$. From any Gröbner basis, one can (easily in practice) compute a partition of the set of unknowns into a maximal subset of transcendental variables $\{U_1, \ldots, U_d\}$ and a minimal subset of algebraic variables $\{X_{d+1}, \ldots, X_n\}$. If W_D is the discriminant variety of $\mathcal{C}$ w.r.t. Π_U, W_D has dimension $< d = \delta$ and one have a full topological description of $\mathcal{C} \setminus \Pi_U^{-1}(W_D)$. We can then apply recursively the same algorithm to decompose $\mathcal{C} \cap W_D$, (and so on, recursively) to get a full description of $\mathcal{C}$. Finally, our framework may also clearly be used for studying simple quantifier elimination problems.

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